# AP Calculus - 5.3 The Fundamental Theorem of Calculus

- Use Part 1 of the Fundamental Theorem
- Use Part 2 of the Fundamental Theorem
- Interpret the integral of a rate of change
- Interpret the integral as an accumulation function

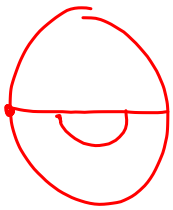
Intro to the fundamental theorem:

Evaluate the following.

Notice what is happening when you are taking the derivative of an integral. Pay particular attention to what ends up happening with the upper and lower bounds of the integral.

$$\frac{d}{dx}\int_{-\pi}^{x} \cos t\, dt = \frac{d}{dx}\left( \sin t \Big|_{-\pi}^{x} \right)$$

$$\frac{d}{dx}\left( \sin x - \sin(-\pi) \right)$$




$$\frac{d}{dx}\left( \sin x - 0 \right) = \frac{d}{dx} \sin x$$

$$= \cos x$$

Notice how when we evaluated sin at (- pi) it was 0 and went away. Maybe that's just a coincidence. Let's see another problem...

$$\frac{d}{dx}\int_{-5}^{x} \sec^2 u\, du = \frac{d}{dx}\left( \tan u \Big|_{-5}^{x} \right)$$

$$= \frac{d}{dx}\left( \tan x - \tan(-5) \right)$$

$$= \frac{d}{dx}\left( \tan x - 3.38 \right)$$

$$= \sec^2 x$$

Now when we integrated we were left with a constant at the end... what happens when you differentiate a constant? Imagine what would happen if you swapped -5 with Any number...?

# Fundamental Theorem of Calculus:

**THEOREM Fundamental Theorem of Calculus, Part 1**

Let $f$ be a function that is continuous on a closed interval $[a, b]$. The function $I$ defined by

$$I(x) = \int_a^x f(t)\,dt$$

$$\boxed{I(x) = \int_a^x f(t)\,dt}$$

has the properties that it is continuous on $[a, b]$ and differentiable on $(a, b)$. Moreover,

$$\boxed{I'(x) = \frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)}$$

for all $x$ in $(a, b)$.

$$I'(x) = \frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$$

The idea here is that regardless of your lower boundary of integration, when your upper bound is a variable and you are differentiating the integral, you will just be left with the integrand where you substitute the upper bound into the variable.

Examples:

**(a)** $\dfrac{d}{dx}\displaystyle\int_0^x \sqrt{t+1}\,dt = \sqrt{x+1}$

**(b)** $\dfrac{d}{dx}\displaystyle\int_2^x \frac{s^3-1}{2s^2+s+1}\,ds = \frac{x^3-1}{2x^2+x+1}$

These are not always super simple problems. Sometimes the upper bound of integration is a function, and the fundamental theorem only works for individual variables... in these cases we will use the chain rule. We are going to refer back to page 3 and 4 of the PDF for 3.1 to refresh.

Find $\frac{d}{dx}\int_4^{3x^2+1}\sqrt{e^t+t}\,dt.$

First, understand the Fund. Theorem only works for a single variable as the upper bound. We are going to use the formal definition for the chain rule, so follow along for this first step.

let $y = \int_4^{3x^2+1}\sqrt{e^t+t}\,dt$

so we are being told to find $\frac{dy}{dx}$

To use the chain rule, and to fix the problem with a function being the upper bound, we will make u = to the upper bound.

$$\frac{dy}{dx}=\frac{dy}{du}*\frac{du}{dx}$$

This is what we are trying to use...

so $u = 3x^2+1$

so ... $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$

$\frac{dy}{dx} = \frac{d}{du}\left(\int_4^u\sqrt{e^t+t}\,dt\right)\cdot\frac{du}{dx}$

Now. the dy/du part looks just like a problem from last slide, except there is a u instead of an x.

$= \sqrt{e^u+u}\cdot\frac{du}{dx}$ $\qquad \frac{du}{dx} = 6x$

$= \sqrt{e^{3x^2+1}+3x^2+1}\cdot 6x$

Working with problems where your lower bound is a variable:

We take advantage of this rule when dealing with integrals like the one below.

- If $a > b$ and if $\int_b^a f(x)\,dx$ exists, then

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

Find $\dfrac{d}{dx}\displaystyle\int_{x^3}^{5} (t^4+1)^{1/3}\,dt.$

$$\frac{d}{dx}\int_5^{x^3} (t^4+1)^{1/3}\,dt$$

← but i'll make the answer flip sign

$$= \left((x^3)^4+1\right)^{1/3} \cdot 3x^2$$

...

$$= 3x^2 \cdot \sqrt[3]{x^{12}+1}$$

$$= -3x^2 \cdot \sqrt[3]{x^{12}+1}$$

## Fundamental Theorem Part 2:

**THEOREM Fundamental Theorem of Calculus, Part 2**

Let $f$ be a function that is continuous on a closed interval $[a, b]$. If $F$ is any antiderivative of $f$ on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

We have already been doing this. We used this theorem on the first slide of notes. This is just the formal definition.

Common Antiderivatives:

| Function $f$ | Antiderivatives $F$ of $f$ |
|---|---|
| $f(x) = 0$ | $F(x) = C$ |
| $f(x) = 1$ | $F(x) = x + C$ |
| $f(x) = x^a, \quad a \neq -1$ | $F(x) = \dfrac{x^{a+1}}{a+1} + C$ |
| $f(x) = x^{-1} = \dfrac{1}{x}$ | $F(x) = \ln\lvert x\rvert + C$ |
| $f(x) = e^x$ | $F(x) = e^x + C$ |
| $f(x) = a^x$ | $F(x) = \dfrac{a^x}{\ln a} + C, \quad a > 0, a \neq 1$ |
| $f(x) = \sin x$ | $F(x) = -\cos x + C$ |
| $f(x) = \cos x$ | $F(x) = \sin x + C$ |
| $f(x) = \sec^2 x$ | $F(x) = \tan x + C$ |

Ex: Find the definite integrals given below:

**(a)** $\displaystyle\int_{-2}^{1} x^2\,dx$ **(b)** $\displaystyle\int_0^{\pi/6} \cos x\,dx$

a) $\left.\dfrac{x^3}{3}\right|_{-2}^{1} = \dfrac{1^3}{3} - \dfrac{(-2)^3}{3}$

b) $\left.\sin x\right|_0^{\pi/6} = \sin\left(\frac{\pi}{6}\right) - \sin(0)$

$= \frac{1}{2}$

Less common antiderivatives of note:

| Function $f$ | Antiderivatives $F$ of $f$ |
|---|---|
| $f(x) = \sec x \tan x$ | $F(x) = \sec x + C$ |
| $f(x) = \csc x \cot x$ | $F(x) = -\csc x + C$ |
| $f(x) = \csc^2 x$ | $F(x) = -\cot x + C$ |
| $f(x) = \dfrac{1}{\sqrt{1-x^2}}, \quad \lvert x\rvert < 1$ | $F(x) = \sin^{-1} x + C$ |
| $f(x) = \dfrac{1}{1+x^2}$ | $F(x) = \tan^{-1} x + C$ |
| $f(x) = \dfrac{1}{x\sqrt{x^2-1}}, \quad \lvert x\rvert > 1$ | $F(x) = \sec^{-1} x + C$ |
| $f(x) = \sinh x$ | $F(x) = \cosh x + C$ |
| $f(x) = \cosh x$ | $F(x) = \sinh x + C$ |

Ex: Find the definite integrals given below:

**(c)** $\displaystyle\int_0^{\sqrt{3}/2} \frac{1}{\sqrt{1-x^2}}\,dx$ **(d)** $\displaystyle\int_1^2 \frac{1}{x}\,dx$

↳ $\sin^{-1} x \Big|_0^{\sqrt{3}/2} = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin^{-1}(0)$

d) $\ln|x| \Big|_1^2 = \ln 2 - \ln 1$

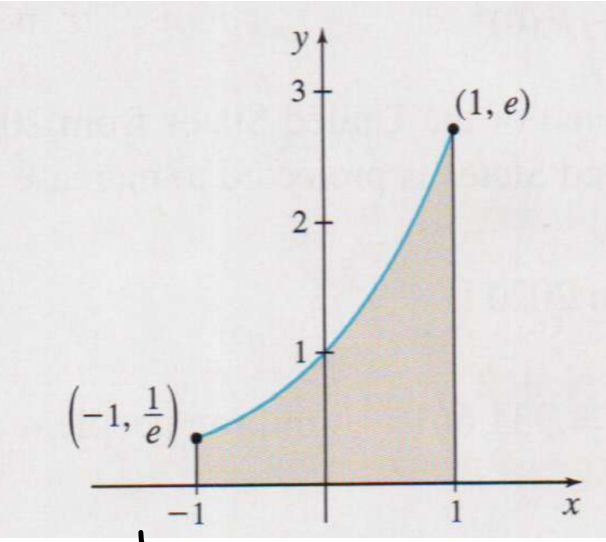
$\approx 0.69$

## Typical definite integral problem:

Ex: Find the area under the graph of f(x) from -1 to 1.

$$f(x) = e^x$$

$$\int_{-1}^{1} e^x \, dx$$




$$e^x \Big|_{-1}^{1} = e^1 - e^{-1} = e - \frac{1}{e}$$

End of Day 1:

Suggested HW Problems:

3-35 Odd

Interpreting the integral of a rate of change:

The integral from $a$ to $b$ of the rate of change of $F$ equals the change in $F$ from $a$ to $b$.

That is,

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

Example: The Population of the US is growing at the rate of $P'(t) = 2.867(1.009)^t$ million people per year, where $t$ is the number of years since 2015.

**(a)** Find $\int_0^5 P'(t)dt$.

**(b)** Interpret $\int_0^5 P'(t)dt$ in the context of the problem.

**(c)** If the population of the United States was 320 million people in 2015, what is the projected population in 2020?

a) $\int_0^5 P'(t)\,dt = \int_0^5 2.867 \cdot \underbrace{1.009^t}_{a}\,dt$

$f'(x) = a^x \quad f(x) = \frac{a^x}{\ln a}$

$= \left( 2.867 \cdot \frac{1.009^t}{\ln 1.009} \right) \Big|_0^5 = \frac{2.867(1.009)^t}{\ln 1.009} \Big|_0^5$

$= \frac{2.867(1.009)^5}{\ln 1.009} - \frac{2.867(1.009)^0}{\ln 1.009} \approx 14.661$

b) The population is projected to increase by approximately 14.661 million from 2015 to 2020 (from 0 to 5)

c) Since the population in 2015 is stated to be 320 million, and we stated that the population will increase by 14.661 million from 2015 to 2020, the population should raise to approximately 334.661 million by 2020.

(Common Free Response Question)

Marginal Profit P'(x) is the rate of change in profit that results from producing and selling one more unit,

$$\longrightarrow \quad \int_{x_1}^{x_2} P'(t)\,dt = P(x_2) - P(x_1)$$

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Ex:

The marginal profit $P'$ in dollars per unit for producing and selling $x$ units of a product is given by

$P'(x) = \sqrt{x}$

Suppose 100 units have been produced and sold. Find the additional profit $P$ from producing and selling 21 more units.

$$\int_{100}^{121} P'(x)\,dx = \int_{100}^{121} \sqrt{x}\,dx$$

$$= \int_{100}^{121} x^{1/2}\,dx = \frac{x^{3/2}}{\frac{3}{2}}\Bigg|_{100}^{121}$$

$$= \frac{2x^{3/2}}{3}\Bigg|_{100}^{121} = \frac{2(121)^{3/2}}{3} - \frac{2(100)^{3/2}}{3}$$

$$\approx 220.667$$

# Interpreting the Integral as an Accumulation Function:

(I will copy and paste the textbook excerpt for absent students, but I'll explain it to present students)

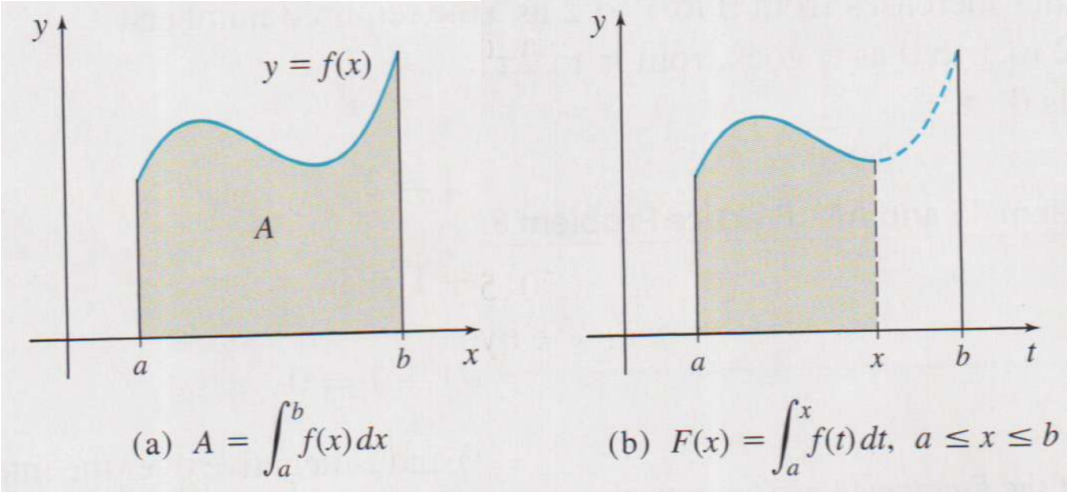



Figure 23

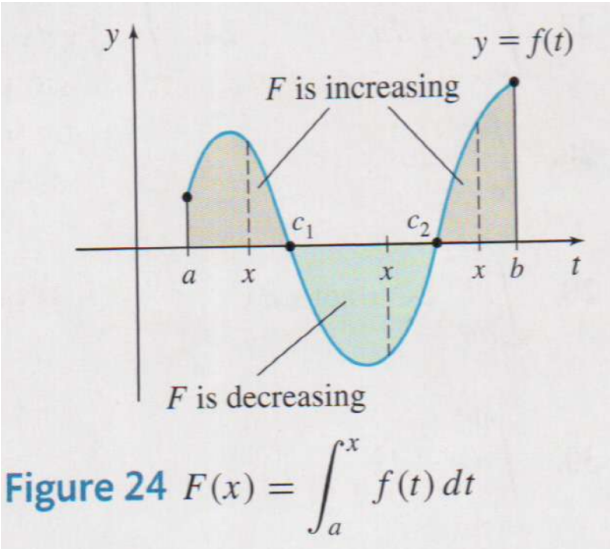



Figure 24 $F(x) = \int_a^x f(t)\,dt$

**DEFINITION** Accumulation Function

Let $f$ be a function that is continuous on the closed interval $[a, b]$. The **accumulation function** $F$ **associated with** $f$ over the interval $[a, b]$ is defined as

$$F(x) = \int_a^x f(t)\,dt \qquad a \le x \le b$$

Excerpt from Textbook over Accumulation.

The figures mentioned are below.

## 4 Interpret the Integral as an Accumulation Function

If $f$ is a function that is continuous and nonnegative on an interval $[a, b]$, then the definite integral $\int_a^b f(x)\,dx$ exists and equals the area $A$ under the graph of $f$ from $a$ to $b$. See Figure 23(a).

Suppose we define the function $F$ to be

$$F(x) = \int_a^x f(t)\,dt \quad a \le x \le b$$

Since the independent variable $x$ of $F$ is the upper limit of integration of the definite integral, we can interpret $F$ as the area under the graph of $f$ from $a$ to the number $x$, as shown in Figure 23(b). Notice that when $x = a$, then $F(a) = \int_a^a f(t)\,dt = 0$ and when $x = b$, then $F(b) = \int_a^b f(t)dt = A$. Notice also that as $x$ increases from $a$ to $b$, the function $F$ also increases, accumulating more area as $x$ goes from $a$ to $b$. It is for this reason we can describe $F(x) = \int_a^x f(t)dt$ as an *accumulation function*.

Now suppose $f$ is not required to be nonnegative on the interval $[a, b]$. Then $f$ may be nonnegative on some subinterval(s) of $[a, b]$ and negative on others. See Figure 24. We examine how this affects $F(x) = \int_a^x f(t)dt$.

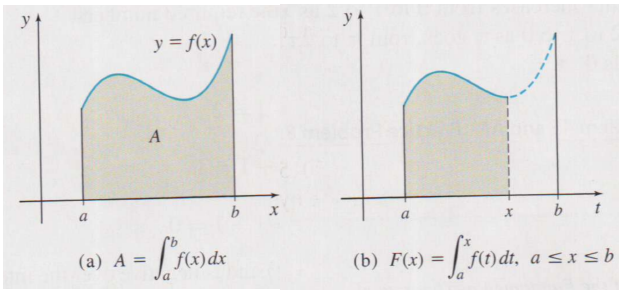
On the interval $[a, c_1]$, the function $f$ is nonnegative. As $x$ increases from $a$ to $c_1$, $F(x) = \int_a^x f(t)dt$ accumulates value and is increasing. On the interval $[c_1, c_2]$, $f(t) \le 0$. As a result, the Riemann sums $\sum_{i=1}^{n} f(u_i)\Delta t_i$ for $f$ on the interval $[c_1, c_2]$ are negative.

So as $x$ increases from $a$ to $c_2$, the accumulation function $F$ increases from $a$ to $c_1$ and then starts to decrease as $x$ goes from $c_1$ to $c_2$, possibly becoming negative. Then from $c_2$ to $b$, the accumulation function $F$ starts to increase again. The net accumulation from $a$ to $b$, given by $\int_a^b f(t)dt$, can be a positive number, a negative number, or zero.

**DEFINITION** Accumulation Function

Let $f$ be a function that is continuous on the closed interval $[a, b]$. The **accumulation function $F$ associated with $f$** over the interval $[a, b]$ is defined as

$$F(x) = \int_a^x f(t)\,dt \qquad a \le x \le b$$




(a) $A = \int_a^b f(x)dx$ (b) $F(x) = \int_a^x f(t)dt,\ a \le x \le b$

Figure 23

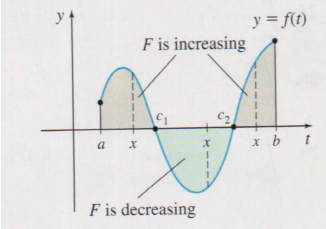
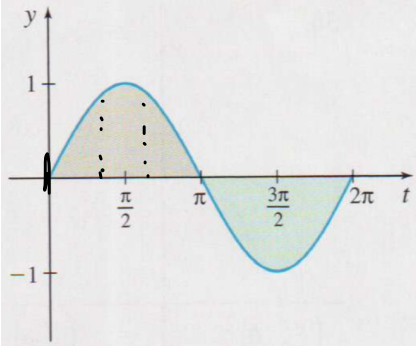



Figure 24 $F(x) = \int_a^x f(t)\,dt$

Ex:




The graph of $f(t) = \sin t, 0 \le t \le 2\pi$ is shown in Figure 25.

**(a)** Find the accumulation function $F$ associated with $f$ over the interval $[0, 2\pi]$.

**(b)** Find $F\left(\frac{\pi}{2}\right)$, $F(\pi)$, $F\left(\frac{3\pi}{2}\right)$, and $F(2\pi)$.

**(c)** Interpret the results found in (b).

$$F(t) = \int_0^x \sin t \, dt$$

$$-\cos t \Big|_0^x = -\cos x - (-\cos 0)$$

$$= -\cos x + 1$$

$$F\left(\frac{\pi}{2}\right) = -\cos\frac{\pi}{2} + 1$$

$$= -0 + 1 = 1$$

$$F(\pi) = -\cos(\pi) + 1$$

$$= --1 + 1$$

$$= +1 + 1$$

$$= 2$$

$$F\left(\frac{3\pi}{2}\right) = -\cos\left(\frac{3\pi}{2}\right) + 1$$

$$= -0 + 1$$

$$= 1$$

$$F(2\pi) = -\cos(2\pi) + 1$$

$$= -1 + 1$$

$$= 0$$

End of Day 2:

Suggested HW Problems:

37-45 odd

End of Day 3:

End of Lesson:

Suggested HW Problems:

59, 61, 63, 67

AP Practice Problems