EXAMPLE 8 Expressing a Limit in Terms of e

Express each limit in terms of the number e:

(a) $\lim_{h\to 0}(1+2h)^{1/h}$ **(b)** $\lim_{n\to\infty}\left(1+\frac{3}{n}\right)^{2n}$

Solution **(a)** This limit resembles $\lim_{h\to 0}(1+h)^{1/h}$, and with some manipulation, it can be expressed in terms of $\lim_{h\to 0}(1+h)^{1/h}$.

$$(1+2h)^{1/h} = [(1+2h)^{1/(2h)}]^2$$

Now let $k = 2h$, and note that $h \to 0$ is equivalent to $2h = k \to 0$. So,

$$\lim_{h\to 0}(1+2h)^{1/h} = \lim_{k\to 0}\left[(1+k)^{1/k}\right]^2 = \left[\lim_{k\to 0}(1+k)^{1/k}\right]^2 \underset{(2)}{=} e^2$$

(b) This limit resembles $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$. We rewrite it as follows:

$$\left(1+\frac{3}{n}\right)^{2n} = \left[\left(1+\frac{3}{n}\right)^{2n}\right]^{3/3} = \left[\left(1+\frac{3}{n}\right)^{n/3}\right]^6$$

Let $k = \frac{n}{3}$. Since $n \to \infty$ is equivalent to $\frac{n}{3} = k \to \infty$, we find that

$$\lim_{n\to\infty}\left(1+\frac{3}{n}\right)^{2n} = \lim_{k\to\infty}\left[\left(1+\frac{1}{k}\right)^k\right]^6 = \left[\lim_{k\to\infty}\left(1+\frac{1}{k}\right)^k\right]^6 = e^6 \quad ■$$

NOW WORK Problem 77.

The number e occurs in many applications. For example, in finance, the number e is used to find **continuously compounded interest**. See the discussion preceding Problem 89.

3.4 Assess Your Understanding

Concepts and Vocabulary

1. $\frac{d}{dx}\ln x =$ ______.
2. *True or False* $\frac{d}{dx}x^e = e\,x^{e-1}$.
3. *True or False* $\frac{d}{dx}\ln[x\sin^2 x] = \frac{d}{dx}\ln x \cdot \frac{d}{dx}\ln\sin^2 x$.
4. *True or False* $\frac{d}{dx}\ln\pi = \frac{1}{\pi}$.
5. $\frac{d}{dx}\ln|x| =$ ____ for all $x \neq 0$.
6. $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n =$ ______.

Skill Building

In Problems 7–44, find the derivative of each function.

7. $f(x) = 5\ln x$
8. $f(x) = -3\ln x$
9. $s(t) = \log_2 t$ (PAGE 248)
10. $s(t) = \log_3 t$
11. $g(x) = (\cos x)(\ln x)$
12. $g(x) = (\sin x)(\ln x)$
13. $F(x) = \ln(3x)$
14. $F(x) = \ln\frac{x}{2}$
15. $s(t) = \ln(e^t - e^{-t})$
16. $s(t) = \ln(e^{at} + e^{-at})$
17. $f(x) = x\ln(x^2+4)$ (PAGE 249)
18. $f(x) = x\ln(x^2+5x+1)$
19. $f(x) = x^2\log_5(3x+5)$
20. $f(x) = x^3\log_4(2x+3)$

21. $f(x) = \frac{1}{2}\ln\frac{1+x}{1-x}$

22. $f(x) = \frac{1}{2}\ln\frac{1+x^2}{1-x^2}$

23. $f(x) = \ln(\ln x)$

24. $f(x) = \ln\left(\ln\frac{1}{x}\right)$

25. $g(x) = \ln\frac{x}{\sqrt{x^2+1}}$

26. $g(x) = \ln\frac{4x^3}{\sqrt{x^2+4}}$

PAGE 249 **27.** $f(x) = \ln\frac{(x^2+1)^2}{x\sqrt{x^2-1}}$

28. $f(x) = \ln\frac{x\sqrt{3x-1}}{(x^2+1)^3}$

29. $F(\theta) = \ln(\sin\theta)$

30. $F(\theta) = \ln(\cos\theta)$

31. $g(x) = \ln\left(x+\sqrt{x^2+4}\right)$

32. $g(x) = \ln\left(\sqrt{x+1}+\sqrt{x}\right)$

33. $f(x) = \log_2(1+x^2)$

34. $f(x) = \log_2(x^2-1)$

35. $f(x) = \tan^{-1}(\ln x)$

36. $f(x) = \sin^{-1}(\ln x)$

37. $s(t) = \ln(\tan^{-1} t)$

38. $s(t) = \ln(\sin^{-1} t)$

39. $G(x) = (\ln x)^{1/2}$

40. $G(x) = (\ln x)^{-1/2}$

41. $f(\theta) = \sin(\ln\theta)$

42. $f(\theta) = \cos(\ln\theta)$

43. $g(x) = (\log_3 x)^{3/2}$

44. $g(x) = (\log_2 x)^{4/3}$

In Problems 45–50, use implicit differentiation to find $y' = \frac{dy}{dx}$.

45. $x\ln y + y\ln x = 2$

46. $\frac{\ln y}{x} + \frac{\ln x}{y} = 2$

47. $\ln(x^2+y^2) = x+y$

48. $\ln(x^2-y^2) = x-y$

49. $\ln\frac{y}{x} = y$

50. $\ln\frac{y}{x} - \ln\frac{x}{y} = 1$

In Problems 51–72, use logarithmic differentiation to find y′. Assume that the variable is restricted so that all arguments of logarithm functions are positive.

PAGE 250 **51.** $y = (x^2+1)^2(2x^3-1)^4$

52. $y = (3x^2+4)^3(x^2+1)^4$

53. $y = \frac{x^2(x^3+1)}{\sqrt{x^2+1}}$

54. $y = \frac{\sqrt{x}(x^3+2)^2}{\sqrt[3]{3x+4}}$

55. $y = \frac{x\cos x}{(x^2+1)^3\sin x}$

56. $y = \frac{x\sin x}{(1+e^x)^3\cos x}$

PAGE 251 **57.** $y = (3x)^x$

58. $y = (x-1)^x$

59. $y = x^{\ln x}$

60. $y = (2x)^{\ln x}$

61. $y = x^{x^2}$

62. $y = (3x)^{\sqrt{x}}$

63. $y = x^{e^x}$

64. $y = (x^2+1)^{e^x}$

65. $y = x^{\sin x}$

66. $y = x^{\cos x}$

67. $y = (\sin x)^x$

68. $y = (\cos x)^x$

69. $y = (\sin x)^{\cos x}$

70. $y = (\sin x)^{\tan x}$

71. $x^y = 4$

72. $y^x = 10$

In Problems 73–76, find an equation of the tangent line to the graph of y = f(x) at the given point.

73. $y = \ln(5x)$ at $\left(\frac{1}{5}, 0\right)$

74. $y = x\ln x$ at $(1, 0)$

PAGE 251 **75.** $y = \frac{x^2\sqrt{3x-2}}{(x-1)^2}$ at $(2, 8)$

76. $y = \frac{x(\sqrt[3]{x}+1)^2}{\sqrt{x+1}}$ at $(8, 24)$

In Problems 77–80, express each limit in terms of e.

PAGE 253 **77.** $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{2n}$

78. $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n/2}$

79. $\lim_{n\to\infty}\left(1+\frac{1}{3n}\right)^{n}$

80. $\lim_{n\to\infty}\left(1+\frac{4}{n}\right)^{n}$

Applications and Extensions

81. Find $\frac{d^{10}}{dx^{10}}(x^9\ln x)$.

82. If $f(x) = \ln(x-1)$, find $f^{(n)}(x)$.

83. If $y = \ln(x^2+y^2)$, find the value of $\frac{dy}{dx}$ at the point $(1, 0)$.

84. If $f(x) = \tan\left(\ln x - \frac{1}{\ln x}\right)$, find $f'(e)$.

85. Find y' if $y = x^x$, $x > 0$, by using $y = x^x = e^{\ln x^x}$ and the Chain Rule.

86. If $y = \ln(kx)$, where $x > 0$ and $k > 0$ is a constant, show that $y' = \frac{1}{x}$.

In Problems 87 and 88, find y′. Assume that a is a constant.

87. $y = x\tan^{-1}\frac{x}{a} - \frac{1}{2}a\ln(x^2+a^2), \quad a \neq 0$

88. $y = x\sin^{-1}\frac{x}{a} + a\ln\sqrt{a^2-x^2}, \quad |a| > |x|, \quad a \neq 0$

Continuously Compounded Interest *In Problems 89 and 90, use the following discussion:*

Suppose an initial investment, called the ***principal*** *P, earns an annual rate of interest r, which is compounded n times per year. The interest earned on the principal P in the first compounding period is* $P\left(\frac{r}{n}\right)$, *and the resulting amount A of the investment after one compounding period is* $A = P + P\left(\frac{r}{n}\right) = P\left(1+\frac{r}{n}\right)$. *After k compounding periods, the amount A of the investment is* $A = P\left(1+\frac{r}{n}\right)^k$. *Since in t years there are nt compounding periods, the amount A after t years is*

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

When interest is compounded so that after t years the accumulated amount is $A = \lim_{n\to\infty} P\left(1+\frac{r}{n}\right)^{nt}$, *the interest is said to be* ***compounded continuously***.

89. **(a)** Show that if the annual rate of interest r is compounded continuously, then the amount A after t years is $A = Pe^{rt}$, where P is the initial investment.

(b) If an initial investment of $P = \$5000$ earns 2% interest compounded continuously, how much is the investment worth after 10 years?

(c) How long does it take an investment of \$10,000 to double if it is invested at 2.4% compounded continuously?

(d) Show that the rate of change of A with respect to t when the interest rate r is compounded continuously is $\frac{dA}{dt} = rA$.

90. A bank offers a certificate of deposit (CD) that matures in 10 years with a rate of interest of 3% compounded continuously. (See Problem 89.) Suppose you purchase such a CD for \$2000 in your IRA.

(a) Write an equation that gives the amount A in the CD as a function of time t in years.

(b) How much is the CD worth at maturity?

(c) What is the rate of change of the amount A at $t = 3$? At $t = 5$? At $t = 8$?

(d) Explain the results found in (c).

91. Sound Level of a Leaf Blower The loudness L, measured in decibels (dB), of a sound of intensity I is defined as $L(x) = 10 \log \dfrac{I(x)}{I_0}$, where x is the distance in meters from the source of the sound and $I_0 = 10^{-12}$ W/m^2 is the least intense sound that a human ear can detect. The intensity I is defined as the power P of the sound wave divided by the area A on which it falls. If the wave spreads out uniformly in all directions, that is, if it is spherical, the surface area is $A(x) = 4\pi x^2$ m^2, and $I(x) = \dfrac{P}{4\pi x^2}$ W/m^2.

(a) If you are 2.0 m from a noisy leaf blower and are walking away from it, at what rate is the loudness L changing with respect to distance x?

(b) Interpret the sign of your answer.

92. Show that $\ln x + \ln y = 2x$ is equivalent to $xy = e^{2x}$. Use this equation to find y'. Compare this result to the solution found in Example 1(c).

93. If $\ln T = kt$, where k is a constant, show that $\dfrac{dT}{dt} = kT$.

94. Graph $y = \left(1 + \dfrac{1}{x}\right)^x$ and $y = e$ on the same set of axes. Explain how the graph supports the fact that $\lim\limits_{n\to\infty} \left(1 + \dfrac{1}{n}\right)^n = e$.

95. Power Rule for Functions Show that if u is a function of x that is differentiable and a is a real number, then

$$\frac{d}{dx}[u(x)]^a = a[u(x)]^{a-1}u'(x)$$

provided u^a and u^{a-1} are defined.
[*Hint:* Let $|y| = |[u(x)]^a|$ and use logarithmic differentiation.]

96. Show that the tangent lines to the graphs of the family of parabolas $f(x) = -\dfrac{1}{2}x^2 + k$ are always perpendicular to the tangent lines to the graphs of the family of natural logarithms $g(x) = \ln(bx) + c$, where $b > 0$, k, and c are constants.

Source: Mathematics students at Millikin University, Decatur, Illinois.

97. Find the derivative of $y = \ln|x|$ by writing $y = \ln\sqrt{x^2}$ and using the Chain Rule.

Challenge Problems

98. Show that $2x - \ln(3 + 6e^x + 3e^{2x}) = C - 2\ln(1 + e^{-x})$ for some constant C.

99. If f and g are differentiable functions, and if $f(x) > 0$, show that

$$\frac{d}{dx} f(x)^{g(x)} = g(x)f(x)^{g(x)-1}f'(x) + f(x)^{g(x)}[\ln f(x)]g'(x)$$

AP® Practice Problems

PAGE 249 **1.** If $f(x) = e^x + x^2$, find $\dfrac{d}{dx}[f(\ln x)]$.

(A) $\dfrac{1 + 2\ln x}{x}$ (B) $1 + \dfrac{2\ln x}{x}$

(C) $1 + \dfrac{2}{x^2}$ (D) $x + 2x\ln x$

PAGE 249 **2.** $\dfrac{d}{dx}(x^2 e^{\ln x^3}) =$

(A) $2x + 3x^2$ (B) $5x^3$ (C) $5x^4$ (D) $6x^4$

PAGE 249 **3.** If $h(x) = \ln(x^2 + 4)$, then $h'(x) =$

(A) $\left|\dfrac{2x}{x^2+4}\right|$ (B) $\dfrac{2x}{x^2+4}$

(C) $\dfrac{x^2}{x^2+4}$ (D) $\dfrac{1}{x^2+4}$

PAGE 248 **4.** Find the rate of change of y with respect to x when $x = 1$ if $\ln(xy) = x$.

(A) e (B) 0 (C) 1 (D) $e - 1$

PAGE 249 **5.** If $f(x) = \ln(e^{x^2-3x})$, then $f'(x) =$

(A) $\dfrac{1}{e^{x^2-3x}}$ (B) $\dfrac{2x-3}{e^{x^2-3x}}$

(C) $x^2 - 3x$ (D) $2x - 3$

PAGE 251 **6.** Find the slope of the tangent line to the graph of $y = \ln(\sec^2 x)$ at $x = \dfrac{\pi}{4}$.

(A) 2 (B) $\dfrac{\sqrt{2}}{2}$ (C) $\sqrt{2}$ (D) $2\sqrt{2}$

PAGE 249 **7.** $\dfrac{d}{dx}\ln\left|\sin\dfrac{\pi}{x}\right| =$

(A) $\cot\dfrac{\pi}{x}$ (B) $-\dfrac{\pi}{x^2}\csc\dfrac{\pi}{x}$

(C) $-\dfrac{\pi}{x^2}\cot\dfrac{\pi}{x}$ (D) $\dfrac{\pi}{x}\cot\dfrac{\pi}{x}$

PAGE 251 **8.** Find $\dfrac{d}{dx}(x^4+2)^x$.

(A) $4x^4(x^4+2)^{x-1}$

(B) $\dfrac{4x^4}{x^4+2}+\ln(x^4+2)$

(C) $(x^4+2)^x\left[\dfrac{4x^4}{x^4+2}+\ln(x^4+2)\right]$

(D) $(x^4+2)^x\ln(x^4+2)$

PAGE 249 **9.** Find $\dfrac{d^2y}{dx^2}$ for $y=\ln(x\sqrt{x})$.

(A) $-\dfrac{3}{2x^2}$ (B) $\dfrac{3}{2x^2}$ (C) $\dfrac{3}{2x}$ (D) $-\dfrac{3}{x^3}$

PAGE 251 **10.** Solve Problem 75 on page 254 using technology.

PAGE 248 **11.** $\dfrac{d}{dx}\dfrac{\log_2 x}{x}=$

(A) $\dfrac{1-\log_2 x}{x^2}$ (B) $-\dfrac{\ln 2+\log_2 x}{x^2}$

(C) $\dfrac{\ln 2-\log_2 x}{x^2\ln 2}$ (D) $\dfrac{1-\ln 2\log_2 x}{x^2\ln 2}$

PAGE 248 **12.** If $f(x)=x\ln x$, then $f'(x)=$

(A) $x+\ln x$ (B) 1 (C) $1+\ln x$ (D) $\dfrac{1}{x}+\ln x$

PAGE 249 **13.** Suppose $g(x)=\ln(f(x))$, where $f(x)>0$ for all real numbers and f is differentiable for all real numbers. If $f(4)=2$ and $f'(4)=-\dfrac{1}{5}$, find $g'(4)$. Show the computations that lead to the answer.

3.5 Differentials; Linear Approximations; Newton's Method

OBJECTIVES ***When you finish this section, you should be able to:***

1. **Find the differential of a function and interpret it geometrically (p. 256)**
2. **Find the linear approximation to a function (p. 258)**
3. **Use differentials in applications (p. 259)**
4. **Use Newton's Method to approximate a real zero of a function (p. 260)**

1 Find the Differential of a Function and Interpret It Geometrically

Recall that for a differentiable function $y=f(x)$, the derivative can be defined as

$$\frac{dy}{dx}=f'(x)=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

That is, for Δx sufficiently close to 0, we can make $\dfrac{\Delta y}{\Delta x}$ as close as we please to $f'(x)$. This can be expressed as

$$\frac{\Delta y}{\Delta x}\approx f'(x)\qquad\text{when}\qquad \Delta x\approx 0\quad \Delta x\neq 0$$

or, since $\Delta x\neq 0$, as

$$\boxed{\Delta y\approx f'(x)\Delta x\qquad\text{when}\qquad \Delta x\approx 0\quad \Delta x\neq 0}\tag{1}$$

DEFINITION

Let $y=f(x)$ be a differentiable function and let Δx denote a change in x.

The **differential of *x***, denoted dx, is defined as $dx=\Delta x\neq 0$.

The **differential of *y***, denoted dy, is defined as

$$\boxed{dy=f'(x)dx}\tag{2}$$