To determine the behavior to the right of $x = \dfrac{5}{2}$, we find the right-hand limit.

$$\lim_{x \to \frac{5}{2}^+} R(x) = \lim_{x \to \frac{5}{2}^+} \left[\frac{3(x+2)}{(2x-5)} \right] = 3 \lim_{x \to \frac{5}{2}^+} \frac{x+2}{2x-5} = \infty$$

As x approaches $\dfrac{5}{2}$ from the right, R becomes unbounded in the positive direction. The graph of R has a vertical asymptote on the right at $x = \dfrac{5}{2}$.

Next we consider $\lim\limits_{x \to 2} R(x)$.

$$\lim_{x \to 2} R(x) = \lim_{x \to 2} \frac{3(x-2)(x+2)}{(2x-5)(x-2)} = \lim_{x \to 2} \frac{3(x+2)}{2x-5} = \frac{3(2+2)}{2 \cdot 2 - 5} = \frac{12}{-1} = -12$$

Since the limit is not infinite, the function R does not have a vertical asymptote at 2. Since 2 is not in the domain of R, the graph of R has a **hole** at the point $(2, -12)$.

To check for horizontal asymptotes, we find the limits at infinity.

$$\lim_{x \to \infty} R(x) = \lim_{x \to \infty} \frac{3x^2 - 12}{2x^2 - 9x + 10} \underset{(2)}{=} \lim_{x \to \infty} \frac{3x^2}{2x^2} = \lim_{x \to \infty} \frac{3}{2} = \frac{3}{2}$$

$$\lim_{x \to -\infty} R(x) = \lim_{x \to -\infty} \frac{3x^2 - 12}{2x^2 - 9x + 10} \underset{(2)}{=} \lim_{x \to -\infty} \frac{3x^2}{2x^2} = \lim_{x \to -\infty} \frac{3}{2} = \frac{3}{2}$$

The line $y = \dfrac{3}{2}$ is a horizontal asymptote of the graph of R for x unbounded in the negative direction and for x unbounded in the positive direction. ∎

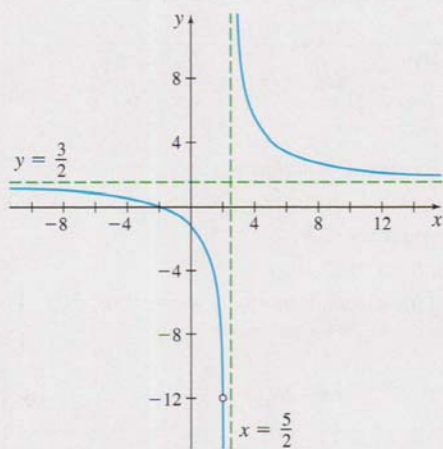The graph of R and its asymptotes are shown in Figure 64. Notice the hole in the graph at the point $(2, -12)$.

Figure 64 $R(x) = \dfrac{3x^2 - 12}{2x^2 - 9x + 10}$

NOW WORK   Problem **69** and AP® Practice Problems **8** and **10**.

## 1.5 Assess Your Understanding

Concepts and Vocabulary

1. *True or False* ∞ is a number.

2. (a) $\lim\limits_{x \to 0^-} \dfrac{1}{x} = $ _____ ; (b) $\lim\limits_{x \to 0^+} \dfrac{1}{x} = $ _____ ;

   (c) $\lim\limits_{x \to 0^+} \ln x = $ _____

3. *True or False*   The graph of a rational function has a vertical asymptote at every number x at which the function is not defined.

4. If $\lim\limits_{x \to 4} f(x) = \infty$, then the line $x = 4$ is a(n) _____ asymptote of the graph of f.

5. (a) $\lim\limits_{x \to \infty} \dfrac{1}{x} = $ _____ ; (b) $\lim\limits_{x \to \infty} \dfrac{1}{x^2} = $ _____ ;

   (c) $\lim\limits_{x \to \infty} \ln x = $ _____

6. *True or False* $\lim\limits_{x \to -\infty} 5 = 0$.

7. (a) $\lim\limits_{x \to -\infty} e^x = $ _____ ; (b) $\lim\limits_{x \to \infty} e^x = $ _____ ; (c) $\lim\limits_{x \to \infty} e^{-x} = $ _____

8. *True or False*   The graph of a function can have at most two horizontal asymptotes.

Skill Building

*In Problems 9–16, use the accompanying graph of $y = f(x)$.*

9. Find $\lim\limits_{x \to \infty} f(x)$.

10. Find $\lim\limits_{x \to -\infty} f(x)$.

11. Find $\lim\limits_{x \to -1^-} f(x)$.

12. Find $\lim\limits_{x \to -1^+} f(x)$.

13. Find $\lim\limits_{x \to 3^-} f(x)$.

14. Find $\lim\limits_{x \to 3^+} f(x)$.

15. Identify all vertical asymptotes.

16. Identify all horizontal asymptotes.
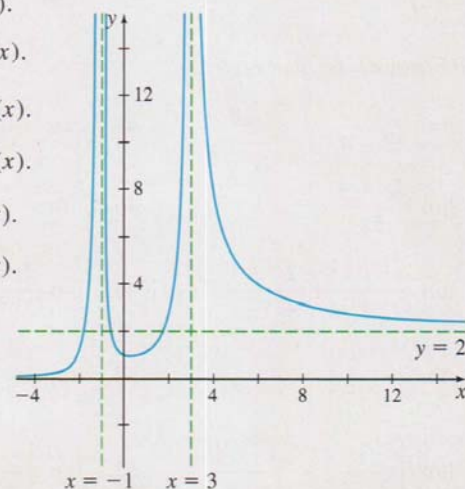
In Problems 17–26, use the graph below of $y = f(x)$.

17. Find $\lim\limits_{x \to \infty} f(x)$.

18. Find $\lim\limits_{x \to -\infty} f(x)$.

19. Find $\lim\limits_{x \to -3^-} f(x)$.

20. Find $\lim\limits_{x \to -3^+} f(x)$.

21. Find $\lim\limits_{x \to 0^-} f(x)$.

22. Find $\lim\limits_{x \to 0^+} f(x)$.

23. Find $\lim\limits_{x \to 4^-} f(x)$.

24. Find $\lim\limits_{x \to 4^+} f(x)$.

25. Identify all vertical asymptotes.

26. Identify all horizontal asymptotes.
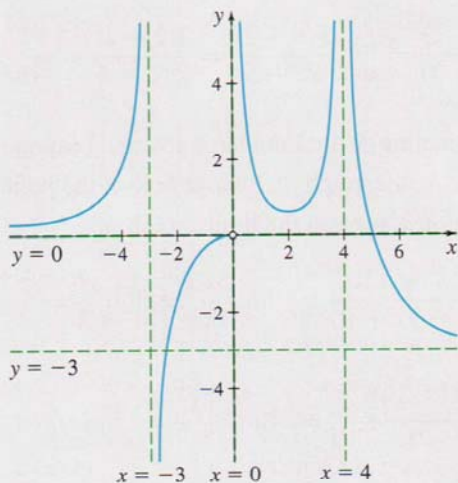
In Problems 27–42, find each limit.

27. $\lim\limits_{x \to 2^-} \dfrac{3x}{x - 2}$

28. $\lim\limits_{x \to -4^+} \dfrac{2x + 1}{x + 4}$

29. $\lim\limits_{x \to 2^+} \dfrac{5}{x^2 - 4}$

30. $\lim\limits_{x \to 1^-} \dfrac{2x}{x^3 - 1}$

31. $\lim\limits_{x \to -1^+} \dfrac{5x + 3}{x(x + 1)}$

32. $\lim\limits_{x \to 0^-} \dfrac{5x + 3}{5x(x - 1)}$

33. $\lim\limits_{x \to -3^-} \dfrac{1}{x^2 - 9}$

34. $\lim\limits_{x \to 2^+} \dfrac{x}{x^2 - 4}$

35. $\lim\limits_{x \to 3} \dfrac{1 - x}{(3 - x)^2}$

36. $\lim\limits_{x \to -1} \dfrac{x + 2}{(x + 1)^2}$

37. $\lim\limits_{x \to \pi^-} \cot x$

38. $\lim\limits_{x \to -\pi/2^-} \tan x$

39. $\lim\limits_{x \to \pi/2^+} \csc(2x)$

40. $\lim\limits_{x \to -\pi/2^-} \sec x$

41. $\lim\limits_{x \to -1^+} \ln(x + 1)$

42. $\lim\limits_{x \to 1^+} \ln(x - 1)$

In Problems 43–60, find each limit.

43. $\lim\limits_{x \to \infty} \dfrac{5}{x^2 + 4}$

44. $\lim\limits_{x \to -\infty} \dfrac{1}{x^2 - 9}$

45. $\lim\limits_{x \to \infty} \dfrac{2x + 4}{5x}$

46. $\lim\limits_{x \to \infty} \dfrac{x + 1}{x}$

47. $\lim\limits_{x \to \infty} \dfrac{x^3 + x^2 + 2x - 1}{x^3 + x + 1}$

48. $\lim\limits_{x \to \infty} \dfrac{2x^2 - 5x + 2}{5x^2 + 7x - 1}$

49. $\lim\limits_{x \to -\infty} \dfrac{x^2 + 1}{x^3 - 1}$

50. $\lim\limits_{x \to \infty} \dfrac{x^2 - 2x + 1}{x^3 + 5x + 4}$

51. $\lim\limits_{x \to \infty} \left[\dfrac{3x}{2x + 5} - \dfrac{x^2 + 1}{4x^2 + 8} \right]$

52. $\lim\limits_{x \to \infty} \left[\dfrac{1}{x^2 + x + 4} - \dfrac{x + 1}{3x - 1} \right]$

53. $\lim\limits_{x \to -\infty} \left[2e^x \left(\dfrac{5x + 1}{3x} \right) \right]$

54. $\lim\limits_{x \to -\infty} \left[e^x \left(\dfrac{x^2 + x - 3}{2x^3 - x^2} \right) \right]$

55. $\lim\limits_{x \to \infty} \dfrac{\sqrt{x} + 2}{3x - 4}$

56. $\lim\limits_{x \to \infty} \dfrac{\sqrt{3x^3} + 2}{x^2 + 6}$

57. $\lim\limits_{x \to \infty} \sqrt{\dfrac{3x^2 - 1}{x^2 + 4}}$

58. $\lim\limits_{x \to \infty} \left(\dfrac{16x^3 + 2x + 1}{2x^3 + 3x} \right)^{2/3}$

59. $\lim\limits_{x \to -\infty} \dfrac{5x^3}{x^2 + 1}$

60. $\lim\limits_{x \to -\infty} \dfrac{x^4}{x - 2}$

In Problems 61–66, find any horizontal or vertical asymptotes of the graph of f.

61. $f(x) = 3 + \dfrac{1}{x}$

62. $f(x) = 2 - \dfrac{1}{x^2}$

63. $f(x) = \dfrac{x^2}{x^2 - 1}$

64. $f(x) = \dfrac{2x^2 - 1}{x^2 - 1}$

65. $f(x) = \dfrac{\sqrt{2x^2 - x + 10}}{2x - 3}$

66. $f(x) = \dfrac{\sqrt[3]{x^2 + 5x}}{x - 6}$

In Problems 67–72, for each rational function R:

(a) Find the domain of R.

(b) Find any horizontal asymptotes of R.

(c) Find any vertical asymptotes of R

(d) Discuss the behavior of the graph at numbers where R is not defined.

67. $R(x) = \dfrac{-2x^2 + 1}{2x^3 + 4x^2}$

68. $R(x) = \dfrac{x^3}{x^4 - 1}$

69. $R(x) = \dfrac{x^2 + 3x - 10}{2x^2 - 7x + 6}$

70. $R(x) = \dfrac{x(x - 1)^2}{(x + 3)^3}$

71. $R(x) = \dfrac{x^3 - 1}{x - x^2}$

72. $R(x) = \dfrac{4x^5}{x^3 - 1}$

Applications and Extensions

In Problems 73 and 74:

(a) *Sketch a graph of a function f that has the given properties.*

(b) *Define a function that describes the graph.*

73. $f(3) = 0$, $\lim\limits_{x \to \infty} f(x) = 1$, $\lim\limits_{x \to -\infty} f(x) = 1$, $\lim\limits_{x \to 1^-} f(x) = \infty$, $\lim\limits_{x \to 1^+} f(x) = -\infty$

74. $f(2) = 0$, $\lim\limits_{x \to \infty} f(x) = 0$, $\lim\limits_{x \to -\infty} f(x) = 0$, $\lim\limits_{x \to 0} f(x) = \infty$, $\lim\limits_{x \to 5^-} f(x) = -\infty$, $\lim\limits_{x \to 5^+} f(x) = \infty$

75. Newton's Law of Cooling Suppose an object is heated to a temperature u_0. Then at time $t = 0$, the object is put into a medium with a constant lower temperature T causing the object to cool. **Newton's Law of Cooling** states that the temperature u of the object at time t is given by $u = u(t) = (u_0 - T)e^{kt} + T$, where $k < 0$ is a constant.

(a) Find $\lim\limits_{t \to \infty} u(t)$. Is this the value you expected? Explain why or why not.

(b) Find $\lim\limits_{t \to 0^+} u(t)$. Is this the value you expected? Explain why or why not.

Source: Submitted by the students of Millikin University.

76. Environment A utility company burns coal to generate electricity. The cost C, in dollars, of removing $p\%$ of the pollutants emitted into the air is

$$C = \frac{70{,}000p}{100 - p}, \qquad 0 \le p < 100$$

Find the cost of removing:

(a) 45% of the pollutants.

(b) 90% of the pollutants.

(c) Find $\lim\limits_{p \to 100^-} C$.

(d) Interpret the answer found in (c).

77. Pollution Control The cost C, in thousands of dollars, to remove a pollutant from a lake is

$$C(x) = \frac{5x}{100 - x}, \qquad 0 \le x < 100$$

where x is the percent of pollutant removed. Find $\lim\limits_{x \to 100^-} C(x)$. Interpret your answer.

78. Population Model A rare species of insect was discovered in the Amazon Rain Forest. To protect the species, entomologists declared the insect endangered and transferred 25 insects to a protected area. The population P of the new colony t days after the transfer is

$$P(t) = \frac{50(1 + 0.5t)}{2 + 0.01t}$$

(a) What is the projected size of the colony after 1 year (365 days)?

(b) What is the largest population that the protected area can sustain? That is, find $\lim\limits_{t \to \infty} P(t)$.

(c) Graph the population P as a function of time t.

(d) Use the graph from (c) to describe the regeneration of the insect population. Does the graph support the answer to (b)?

79. Population of an Endangered Species Often environmentalists capture several members of an endangered species and transport them to a controlled environment where they can produce offspring and regenerate their population. Suppose six American bald eagles are captured, tagged, transported to Montana, and set free. Based on past experience, the environmentalists expect the population to grow according to the model

$$P(t) = \frac{500}{1 + 82.3e^{-0.162t}}$$

where t is measured in years.

(a) If the model is correct, how many bald eagles can the environment sustain? That is, find $\lim\limits_{t \to \infty} P(t)$.

(b) Graph the population P as a function of time t.

(c) Use the graph from (b) to describe the growth of the bald eagle population. Does the graph support the answer to (a)?

80. Hailstones Hailstones typically originate at an altitude of about 3000 meters (m). If a hailstone falls from 3000 m with no air resistance, its speed when it hits the ground would be about 240 meters/second (m/s), which is 540 miles/hour (mi/h)! That would be deadly! But air resistance slows the hailstone considerably. Using a simple model of air resistance, the speed

$v = v(t)$ of a hailstone of mass m as a function of time t is given by $v(t) = \frac{mg}{k}(1 - e^{-kt/m})$ m/s, where $g = 9.8 \text{ m/s}^2$ and k is a constant that depends on the size of the hailstone, its mass, and the conditions of the air. For a hailstone with a diameter $d = 1$ centimeter (cm) and mass m $= 4.8 \times 10^{-4}$ kg, k has been measured to be 3.4×10^{-4} kg/s.

(a) Determine the limiting speed of the hailstone by finding $\lim\limits_{t \to \infty} v(t)$. Express your answer in meters per second and miles per hour, using the fact that $1 \text{ mi/h} \approx 0.447 \text{ m/s}$. This speed is called the **terminal speed** of the hailstone.

(b) Graph $v = v(t)$. Does the graph support the answer to (a)?

81. Damped Harmonic Motion The motion of a spring is given by the function

$$x(t) = 1.2e^{-t/2} \cos t + 2.4e^{-t/2} \sin t$$

where x is the distance in meters from the the equilibrium position and t is the time in seconds.

(a) Graph $y = x(t)$. What is $\lim\limits_{t \to \infty} x(t)$, as suggested by the graph?

(b) Find $\lim\limits_{t \to \infty} x(t)$.

(c) Compare the results of (a) and (b). Is the answer to (b) supported by the graph in (a)?

82. Decomposition of Chlorine in a Pool Under certain water conditions, the free chlorine (hypochlorous acid, HOCl) in a swimming pool decomposes according to the law of uninhibited decay, $C = C(t) = C(0)e^{kt}$, where $C = C(t)$ is the amount (in parts per million, ppm) of free chlorine present at time t (in hours) and k is a negative number that represents the rate of decomposition. After shocking his pool, Ben immediately tested the water and found the concentration of free chlorine to be $C_0 = C(0) = 2.5$ ppm. Twenty-four hours later, Ben tested the water again and found the amount of free chlorine to be 2.2 ppm.

(a) What amount of free chlorine will be left after 72 hours?

(b) When the free chlorine reaches 1.0 ppm, the pool should be shocked again. How long can Ben go before he must shock the pool again?

(c) Find $\lim\limits_{t \to \infty} C(t)$.

(d) Interpret the answer found in (c).

83. Decomposition of Sucrose Reacting with water in an acidic solution at 35°C, the amount A of sucrose ($C_{12}H_{22}O_{11}$) decomposes into glucose ($C_6H_{12}O_6$) and fructose ($C_6H_{12}O_6$) according to the law of uninhibited decay $A = A(t) = A(0)e^{kt}$, where $A = A(t)$ is the amount (in moles) of sucrose present at time t (in minutes) and k is a negative number that represents the rate of decomposition. An initial amount $A_0 = A(0) = 0.40$ mole of sucrose decomposes to 0.36 mole in 30 minutes.

(a) How much sucrose will remain after 2 hours?

(b) How long will it take until 0.10 mole of sucrose remains?

(c) Find $\lim\limits_{t \to \infty} A(t)$.

(d) Interpret the answer found in (c).

84. Macrophotography A camera lens can be approximated by a thin lens. A thin lens of focal length f obeys the thin-lens equation $\frac{1}{f} = \frac{1}{p} + \frac{1}{q}$, where $p > f$ is the distance from the

lens to the object being photographed and q is the distance from the lens to the image formed by the lens. See the figure below. To photograph an object, the object's image must be formed on the photo sensors of the camera, which can only occur if q is positive.

Object   Lens   Photo
                             sensors
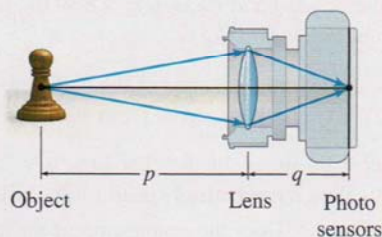
(a) Is the distance q of the image from the lens continuous as the distance of the object being photographed approaches the focal length f of the lens? (*Hint:* First solve the thin-lens equation for q and then find $\lim\limits_{p \to f^+} q$.)

(b) Use the result from (a) to explain why a camera (or any lens) cannot focus on an object placed close to its focal length.

In Problems 85 and 86, find conditions on a, b, c, and d so that the graph of f has no horizontal or vertical asymptotes.

85. $f(x) = \dfrac{ax^3 + b}{cx^4 + d}$ 86. $f(x) = \dfrac{ax + b}{cx + d}$

87. Explain why the following properties are true. Give an example of each.

(a) If n is an even positive integer, then $\lim\limits_{x \to c} \dfrac{1}{(x-c)^n} = \infty$.

(b) If n is an odd positive integer, then $\lim\limits_{x \to c^-} \dfrac{1}{(x-c)^n} = -\infty$.

(c) If n is an odd positive integer, then $\lim\limits_{x \to c^+} \dfrac{1}{(x-c)^n} = \infty$.

88. Explain why a rational function, whose numerator and denominator have no common zeros, will have vertical asymptotes at each point of discontinuity.

89. Explain why a polynomial function of degree 1 or higher cannot have any asymptotes.

90. If P and Q are polynomials of degree m and n, respectively, discuss $\lim\limits_{x \to \infty} \dfrac{P(x)}{Q(x)}$ when:

(a) $m > n$ (b) $m = n$ (c) $m < n$

91. [N] (a) Use a table to investigate $\lim\limits_{x \to \infty} \left(1 + \dfrac{1}{x}\right)^x$.

[CAS] (b) Find $\lim\limits_{x \to \infty} \left(1 + \dfrac{1}{x}\right)^x$.

(c) Compare the results from (a) and (b). Explain the possible causes of any discrepancy.

Challenge Problems

92. $\lim\limits_{x \to \infty} \left(1 + \dfrac{1}{x}\right) = 1$, but $\lim\limits_{x \to \infty} \left(1 + \dfrac{1}{x}\right)^x > 1$. Discuss why the property $\lim\limits_{x \to \infty} [f(x)]^n = \left[\lim\limits_{x \to \infty} f(x)\right]^n$ cannot be used to find the second limit.

93. **Kinetic Energy** At low speeds the kinetic energy K, that is, the energy due to the motion of an object of mass m and speed v, is given by the formula $K = K(v) = \dfrac{1}{2}mv^2$. But this formula is only an approximation to the general formula, and works only for speeds much less than the speed of light, c. The general formula, which holds for all speeds, is

$$K_{gen}(v) = mc^2 \left[\frac{1}{\sqrt{1 - \dfrac{v^2}{c^2}}} - 1\right]$$

(a) As an object is accelerated closer and closer to the speed of light, what does its kinetic energy K_{gen} approach?

(b) What does the result suggest about the possibility of reaching the speed of light?

AP® Practice Problems

[PAGE 138] 1. For $x > 0$, the line $y = 1$ is an asymptote of the graph of a function f. Which of the following statements must be true?

(A) $f(x) \neq 1$ for $x > 0$.   (B) $\lim\limits_{x \to 1} f(x) = \infty$

(C) $\lim\limits_{x \to \infty} f(x) = 1$ (D) $\lim\limits_{x \to -\infty} f(x) = 1$

[PAGE 135] 2. $\lim\limits_{x \to \infty} \dfrac{3x^3 + 4x^2 - x + 10}{2x^4 - x^3 + 2x^2 - 2} =$

(A) -5 (B) 0 (C) $\dfrac{3}{2}$ (D) ∞

[PAGE 135] 3. $\lim\limits_{x \to \infty} \dfrac{5x^3 - x}{8 - x^3} =$

(A) -5 (B) $\dfrac{5}{8}$ (C) 5 (D) ∞

[PAGE 138] 4. Find all the horizontal asymptotes of the graph of $y = \dfrac{2 + 3^x}{4 - 3^x}$.

(A) $y = -1$ only   (B) $y = \dfrac{1}{2}$ only

(C) $y = -1$ and $y = 0$ (D) $y = -1$ and $y = \dfrac{1}{2}$

[PAGE 131] 5. Find all the vertical asymptotes of the graph of

$r(x) = \dfrac{x^2 + 5x + 6}{x^3 - 4x}$.

(A) $x = 0$ and $x = -2$ (B) $x = 0$ and $x = 2$

(C) $x = -2$ and $x = 2$ (D) $x = 0, x = -2$ and $x = 2$

[PAGE 134] 6. $\lim\limits_{x \to -\infty} \dfrac{\sqrt{8x^2 - 4x}}{x + 2} =$

(A) $-\infty$ (B) $-2\sqrt{2}$ (C) 4 (D) $2\sqrt{2}$

7. The graph of which of the following functions has an asymptote of $y = 1$?

(A) $y = \cos x$ (B) $y = \dfrac{x-1}{x}$

(C) $y = e^{-x}$ (D) $y = \ln x$

8. If the graph of $f(x) = \dfrac{ax - b}{x + c}$ has a vertical asymptote $x = -5$ and horizontal asymptote $y = -3$, then $a + c =$

(A) –8   (B) –2   (C) $\dfrac{3}{5}$ (D) 2

9. $\displaystyle\lim_{x \to 1^-} \dfrac{x}{\ln x}$ is

(A) $-\infty$ (B) –1   (C) 1   (D) ∞

10. The function $f(x) = \dfrac{2x}{|x| - 1}$ has

(A) no vertical asymptote and one horizontal asymptote.
(B) one vertical asymptote and one horizontal asymptote.
(C) two vertical asymptotes and one horizontal asymptote.
(D) two vertical asymptotes and two horizontal asymptotes.

11. $\displaystyle\lim_{x \to 2^-} \dfrac{5x + 1}{2x - 4} =$

(A) $-\infty$ (B) $-\dfrac{5}{2}$ (C) $\dfrac{5}{2}$ (D) ∞

1.6 The ε-δ Definition of a Limit

OBJECTIVES *When you finish this section, you should be able to:*

1 Use the ε-δ definition of a limit (p. 145)

Throughout the chapter, we stated that we could be sure a limit was correct only if it was based on the ε-δ definition of a limit. In this section, we examine this definition and how to use it to prove a limit exists, to verify the value of a limit, and to show that a limit does not exist.

Consider the function f defined by

$$f(x) = \begin{cases} 3x + 1 & \text{if } x \neq 2 \\ 10 & \text{if } x = 2 \end{cases}$$

whose graph is given in Figure 65.

As x gets closer to 2, the value $f(x)$ gets closer to 7. If in fact, by taking x close enough to 2, we can make $f(x)$ as close to 7 as we please, then $\displaystyle\lim_{x \to 2} f(x) = 7$.

Suppose we want $f(x)$ to differ from 7 by less than 0.3; that is,

$$-0.3 < f(x) - 7 < 0.3$$
$$6.7 < f(x) < 7.3$$

How close must x be to 2? First, we must require $x \neq 2$ because when $x = 2$, then $f(x) = f(2) = 10$, and we obtain $6.7 < 10 < 7.3$, which is impossible.

Then, when $x \neq 2$,

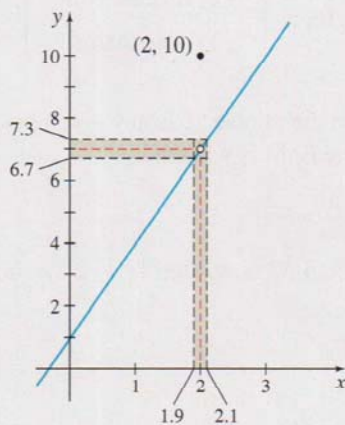$$-0.3 < f(x) - 7 < 0.3$$
$$-0.3 < (3x + 1) - 7 < 0.3$$
$$-0.3 < 3x - 6 < 0.3$$
$$-0.3 < 3(x - 2) < 0.3$$
$$\frac{-0.3}{3} < x - 2 < \frac{0.3}{3}$$
$$-0.1 < x - 2 < 0.1$$
$$|x - 2| < 0.1$$

Figure 65 $f(x) = \begin{cases} 3x + 1 & \text{if } x \neq 2 \\ 10 & \text{if } x = 2 \end{cases}$