The discussion in Example 8 forms the basis of the definition of a limit. We state the definition here, but postpone the details until Section 1.6. It is customary to use the Greek letters $\varepsilon$ (epsilon) and $\delta$ (delta) in the definition, so we call it the *$\varepsilon$-$\delta$ definition of a limit*.

**DEFINITION $\varepsilon$-$\delta$ Definition of a Limit**

Let $f$ be a function defined everywhere in an open interval containing $c$, except possibly at $c$. Then the **limit of the function $f$ as $x$ approaches $c$** is the number $L$, written

$$\boxed{\lim_{x\to c} f(x) = L}$$

if, given any number $\varepsilon > 0$, there is a number $\delta > 0$ so that

$$\boxed{\text{whenever } 0 < |x - c| < \delta \quad \text{then } |f(x) - L| < \varepsilon}$$

Notice in the definition that $f$ is defined everywhere in an open interval containing $c$ except possibly at $c$. If $f$ is defined at $c$ and there is an open interval containing $c$ that contains no other numbers in the domain of $f$, then $\lim_{x\to c} f(x)$ does not exist.

## 1.1 Assess Your Understanding

### Concepts and Vocabulary

1. *Multiple Choice* The limit as $x$ approaches $c$ of a function $f$ is written symbolically as [**(a)** $\lim f(x)$, **(b)** $\lim_{c\to x} f(x)$, **(c)** $\lim_{x\to c} f(x)$]
2. *True or False* The tangent line to the graph of $f$ at a point $P = (c, f(c))$ is the limiting position of the secant lines passing through $P$ and a point $(x, f(x))$, $x \neq c$, as $x$ moves closer to $c$.
3. *True or False* If $f$ is not defined at $x = c$, then $\lim_{x\to c} f(x)$ does not exist.
4. *True or False* The limit $L$ of a function $y = f(x)$ as $x$ approaches the number $c$ depends on the value of $f$ at $c$.
5. *True or False* If $f(c)$ is defined, this suggests that $\lim_{x\to c} f(x)$ exists.
6. *True or False* The limit of a function $y = f(x)$ as $x$ approaches a number $c$ equals $L$ if at least one of the one-sided limits as $x$ approaches $c$ equals $L$.

### Skill Building

*In Problems 7–12, complete each table and investigate the limit.*

7. $\lim_{x\to 1} 2x$

| | $x$ approaches 1 from the left → | | $x$ approaches 1 from the right ← |
|---|---|---|---|
| $x$ | 0.9 0.99 0.999 | → 1 ← | 1.001 1.01 1.1 |
| $f(x) = 2x$ | | | |

8. $\lim_{x\to 2} (x + 3)$

| | $x$ approaches 2 from the left → | | $x$ approaches 2 from the right ← |
|---|---|---|---|
| $x$ | 1.9 1.99 1.999 | → 2 ← | 2.001 2.01 2.1 |
| $f(x) = x + 3$ | | | |

PAGE 81 9. $\lim_{x\to 0} (x^2 + 2)$

| | $x$ approaches 0 from the left → | | $x$ approaches 0 from the right ← |
|---|---|---|---|
| $x$ | −0.1 −0.01 −0.001 | → 0 ← | 0.001 0.01 0.1 |
| $f(x) = x^2 + 2$ | | | |

10. $\lim_{x\to -1} (x^2 - 2)$

| | $x$ approaches −1 from the left → | | $x$ approaches −1 from the right ← |
|---|---|---|---|
| $x$ | −1.1 −1.01 −1.001 | → −1 ← | −0.999 −0.99 −0.9 |
| $f(x) = x^2 - 2$ | | | |

11. $\lim_{x\to -3} \dfrac{x^2 - 9}{x + 3}$

| | $x$ approaches −3 from the left → | | $x$ approaches −3 from the right ← |
|---|---|---|---|
| $x$ | −3.5 −3.1 −3.01 | → −3 ← | −2.99 −2.9 −2.5 |
| $f(x) = \dfrac{x^2 - 9}{x + 3}$ | | | |

12. $\lim_{x\to -1} \dfrac{x^3 + 1}{x + 1}$

| | $x$ approaches −1 from the left → | | $x$ approaches −1 from the right ← |
|---|---|---|---|
| $x$ | −1.1 −1.01 −1.001 | → −1 ← | −0.999 −0.99 −0.9 |
| $f(x) = \dfrac{x^3 + 1}{x + 1}$ | | | |

*In Problems 13–16, use technology to complete the table and investigate the limit.*

PAGE 82 **13.** $\lim\limits_{x\to 0} \dfrac{2-2e^x}{x}$

| | $x$ approaches 0 from the left → | | | | | ← $x$ approaches 0 from the right | | |
|---|---|---|---|---|---|---|---|---|
| $x$ | −0.2 | −0.1 | −0.01 | → 0 ← | 0.01 | 0.1 | 0.2 |
| $f(x) = \dfrac{2-2e^x}{x}$ | | | | | | | |

**14.** $\lim\limits_{x\to 1} \dfrac{\ln x}{x-1}$

| | $x$ approaches 1 from the left → | | | | ← $x$ approaches 1 from the right | | |
|---|---|---|---|---|---|---|---|
| $x$ | 0.9 | 0.99 | 0.999 | → 1 ← | 1.001 | 1.01 | 1.1 |
| $f(x) = \dfrac{\ln x}{x-1}$ | | | | | | | |

PAGE 82 **15.** $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x}$, where $x$ is measured in radians

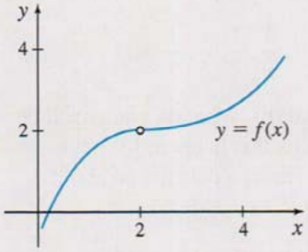
| | $x$ approaches 0 from the left → | | | | ← $x$ approaches 0 from the right | | |
|---|---|---|---|---|---|---|---|
| $x$ (in radians) | −0.2 | −0.1 | −0.01 | → 0 ← | 0.01 | 0.1 | 0.2 |
| $f(x) = \dfrac{1-\cos x}{x}$ | | | | | | | |

**16.** $\lim\limits_{x\to 0} \dfrac{\sin x}{1+\tan x}$, where $x$ is measured in radians

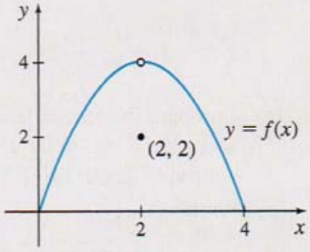
| | $x$ approaches 0 from the left → | | | | ← $x$ approaches 0 from the right | | |
|---|---|---|---|---|---|---|---|
| $x$ (in radians) | −0.2 | −0.1 | −0.01 | → 0 ← | 0.01 | 0.1 | 0.2 |
| $f(x) = \dfrac{\sin x}{1+\tan x}$ | | | | | | | |

*In Problems 17–20, use the graph to investigate (a)* $\lim\limits_{x\to 2^-} f(x)$, *(b)* $\lim\limits_{x\to 2^+} f(x)$, *(c)* $\lim\limits_{x\to 2} f(x)$.

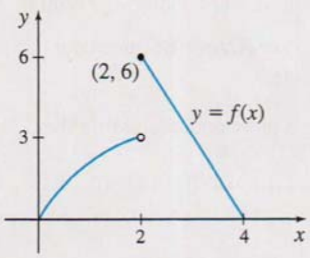
**17.**



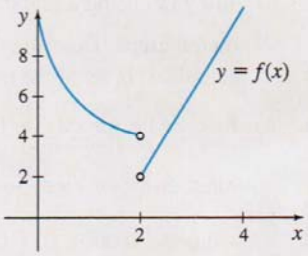

**18.**




**19.**

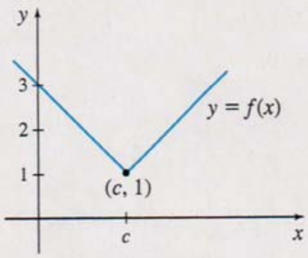



**20.**

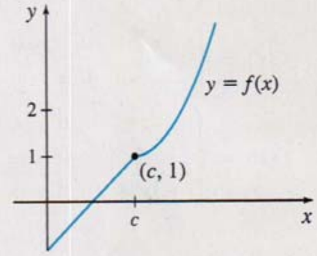



*In Problems 21–28, use the graph to investigate* $\lim\limits_{x\to c} f(x)$. *If the limit does not exist, explain why.*

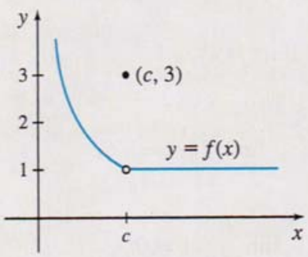
**21.**



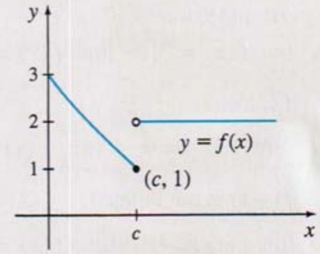

**22.**



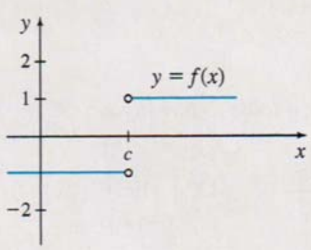

**23.**



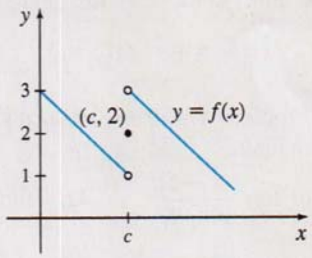

**24.**



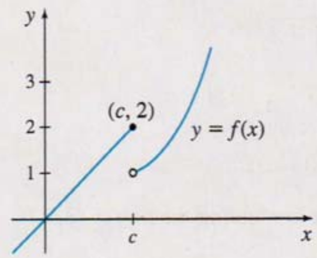

PAGE 84 **25.**



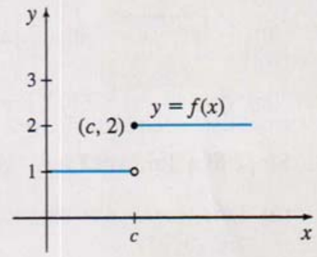

**26.**




**27.**




**28.**




*In Problems 29–36, use a graph to investigate* $\lim\limits_{x\to c} f(x)$ *at the number c.*

**29.** $f(x) = \begin{cases} 2x+5 & \text{if } x \le 2 \\ 4x+1 & \text{if } x > 2 \end{cases}$ at $c = 2$

**30.** $f(x) = \begin{cases} 2x+1 & \text{if } x \le 0 \\ 2x & \text{if } x > 0 \end{cases}$ at $c = 0$

PAGE 84 **31.** $f(x) = \begin{cases} 3x-1 & \text{if } x < 1 \\ 4 & \text{if } x = 1 \\ 4x & \text{if } x > 1 \end{cases}$ at $c = 1$

**32.** $f(x) = \begin{cases} x+2 & \text{if } x < 2 \\ 4 & \text{if } x = 2 \\ x^2 & \text{if } x > 2 \end{cases}$ at $c = 2$

33. $f(x) = \begin{cases} 2x^2 & \text{if } x < 1 \\ 3x^2 - 1 & \text{if } x > 1 \end{cases}$ at $c = 1$

34. $f(x) = \begin{cases} x^3 & \text{if } x < -1 \\ x^2 - 1 & \text{if } x > -1 \end{cases}$ at $c = -1$

35. $f(x) = \begin{cases} x^2 & \text{if } x \le 0 \\ 2x + 1 & \text{if } x > 0 \end{cases}$ at $c = 0$

36. $f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ 2 & \text{if } x = 1 \\ -3x + 2 & \text{if } x > 1 \end{cases}$ at $c = 1$

## Applications and Extensions

*In Problems 37–40, sketch a graph of a function with the given properties. Answers will vary.*

37. $\lim_{x\to 2} f(x) = 3$; $\lim_{x\to 3^-} f(x) = 3$; $\lim_{x\to 3^+} f(x) = 1$; $f(2) = 3$; $f(3) = 1$

38. $\lim_{x\to -1} f(x) = 0$; $\lim_{x\to 2^-} f(x) = -2$; $\lim_{x\to 2^+} f(x) = -2$; $f(-1)$ is not defined; $f(2) = -2$

39. $\lim_{x\to 1} f(x) = 4$; $\lim_{x\to 0^-} f(x) = -1$; $\lim_{x\to 0^+} f(x) = 0$; $f(0) = -1$; $f(1) = 2$

40. $\lim_{x\to 2} f(x) = 2$; $\lim_{x\to -1} f(x) = 0$; $\lim_{x\to 1} f(x) = 1$; $f(-1) = 1$; $f(2) = 3$

*In Problems 41–50, use either a graph or a table to investigate each limit.*

41. $\lim_{x\to 5^+} \dfrac{|x-5|}{x-5}$

42. $\lim_{x\to 5^-} \dfrac{|x-5|}{x-5}$

43. $\lim_{x\to \left(\frac{1}{2}\right)^-} \lfloor 2x \rfloor$

44. $\lim_{x\to \left(\frac{1}{2}\right)^+} \lfloor 2x \rfloor$

45. $\lim_{x\to \left(\frac{2}{3}\right)^-} \lfloor 2x \rfloor$

46. $\lim_{x\to \left(\frac{2}{3}\right)^+} \lfloor 2x \rfloor$

47. $\lim_{x\to 2^+} \sqrt{|x| - x}$

48. $\lim_{x\to 2^-} \sqrt{|x| - x}$

49. $\lim_{x\to 2^+} \sqrt[3]{\lfloor x \rfloor - x}$

50. $\lim_{x\to 2^-} \sqrt[3]{\lfloor x \rfloor - x}$

51. **Slope of a Tangent Line** For $f(x) = 3x^2$:
   (a) Find the slope of the secant line containing the points (2, 12) and (3, 27).
   (b) Find the slope of the secant line containing the points (2, 12) and $(x, f(x))$, $x \neq 2$.
   (c) Create a table to investigate the slope of the tangent line to the graph of $f$ at 2 using the result from (b).
   (d) On the same set of axes, graph $f$, the tangent line to the graph of $f$ at the point (2, 12), and the secant line from (a).

52. **Slope of a Tangent Line** For $f(x) = x^3$:
   (a) Find the slope of the secant line containing the points (2, 8) and (3, 27).
   (b) Find the slope of the secant line containing the points (2, 8) and $(x, f(x))$, $x \neq 2$.
   (c) Create a table to investigate the slope of the tangent line to the graph of $f$ at 2 using the result from (b).
   (d) On the same set of axes, graph $f$, the tangent line to the graph of $f$ at the point (2, 8), and the secant line from (a).

53. **Slope of a Tangent Line** For $f(x) = \frac{1}{2}x^2 - 1$:
   (a) Find the slope $m_{sec}$ of the secant line containing the points $P = (2, f(2))$ and $Q = (2+h, f(2+h))$.
   (b) Use the result from (a) to complete the following table:

| $h$ | $-0.5$ | $-0.1$ | $-0.001$ | $0.001$ | $0.1$ | $0.5$ |
|---|---|---|---|---|---|---|
| $m_{sec}$ | | | | | | |

   (c) Investigate the limit of the slope of the secant line found in (a) as $h \to 0$.
   (d) What is the slope of the tangent line to the graph of $f$ at the point $P = (2, f(2))$?
   (e) On the same set of axes, graph $f$ and the tangent line to $f$ at $P = (2, f(2))$.

54. **Slope of a Tangent Line** For $f(x) = x^2 - 1$:
   (a) Find the slope $m_{sec}$ of the secant line containing the points $P = (-1, f(-1))$ and $Q = (-1+h, f(-1+h))$.
   (b) Use the result from (a) to complete the following table:

| $h$ | $-0.1$ | $-0.01$ | $-0.001$ | $-0.0001$ | $0.0001$ | $0.001$ | $0.01$ | $0.1$ |
|---|---|---|---|---|---|---|---|---|
| $m_{sec}$ | | | | | | | | |

   (c) Investigate the limit of the slope of the secant line found in (a) as $h \to 0$.
   (d) What is the slope of the tangent line to the graph of $f$ at the point $P = (-1, f(-1))$?
   (e) On the same set of axes, graph $f$ and the tangent line to $f$ at $P = (-1, f(-1))$.

PAGE 85 55. (a) Investigate $\lim_{x\to 0} \cos\dfrac{\pi}{x}$ by using a table and evaluating the function $f(x) = \cos\dfrac{\pi}{x}$ at
$$x = -\frac{1}{2}, -\frac{1}{4}, -\frac{1}{8}, -\frac{1}{10}, -\frac{1}{12}, \ldots, \frac{1}{12}, \frac{1}{10}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}.$$
   (b) Investigate $\lim_{x\to 0} \cos\dfrac{\pi}{x}$ by using a table and evaluating the function $f(x) = \cos\dfrac{\pi}{x}$ at
$$x = -1, -\frac{1}{3}, -\frac{1}{5}, -\frac{1}{7}, -\frac{1}{9}, \ldots, \frac{1}{9}, \frac{1}{7}, \frac{1}{5}, \frac{1}{3}, 1.$$
   (c) Compare the results from (a) and (b). What do you conclude about the limit? Why do you think this happens? What is your view about using a table to draw a conclusion about limits?
   (d) Use technology to graph $f$. Begin with the $x$-window $[-2\pi, 2\pi]$ and the $y$-window $[-1, 1]$. If you were finding $\lim_{x\to 0} f(x)$ using a graph, what would you conclude? Zoom in on the graph. Describe what you see. (*Hint*: Be sure your calculator is set to the radian mode.)

56. (a) Investigate $\lim_{x\to 0} \cos\dfrac{\pi}{x^2}$ by using a table and evaluating the function $f(x) = \cos\dfrac{\pi}{x^2}$ at $x = -0.1, -0.01, -0.001, -0.0001, 0.0001, 0.001, 0.01, 0.1$.

(b) Investigate $\lim_{x \to 0} \cos \frac{\pi}{x^2}$ by using a table and evaluating the function $f(x) = \cos \frac{\pi}{x^2}$ at

$$x = -\frac{2}{3}, -\frac{2}{5}, -\frac{2}{7}, -\frac{2}{9}, \ldots, \frac{2}{9}, \frac{2}{7}, \frac{2}{5}, \frac{2}{3}.$$

(c) Compare the results from (a) and (b). What do you conclude about the limit? Why do you think this happens? What is your view about using a table to draw a conclusion about limits?

(d) Use technology to graph $f$. Begin with the $x$-window $[-2\pi, 2\pi]$ and the $y$-window $[-1, 1]$. If you were finding $\lim_{x \to 0} f(x)$ using a graph, what would you conclude? Zoom in on the graph. Describe what you see. (*Hint*: Be sure your calculator is set to the radian mode.)

PAGE 85 **57.** (a) Use a table to investigate $\lim_{x \to 2} \frac{x-8}{2}$.

(b) How close must $x$ be to 2, so that $f(x)$ is within 0.1 of the limit?

(c) How close must $x$ be to 2, so that $f(x)$ is within 0.01 of the limit?

**58.** (a) Use a table to investigate $\lim_{x \to 2}(5 - 2x)$.

(b) How close must $x$ be to 2, so that $f(x)$ is within 0.1 of the limit?

(c) How close must $x$ be to 2, so that $f(x)$ is within 0.01 of the limit?

**59. First-Class Mail** As of April 2016, the U.S. Postal Service charged \$0.47 postage for first-class letters weighing up to and including 1 ounce, plus a flat fee of \$0.21 for each additional or partial ounce up to and including 3.5 ounces. First-class letter rates do not apply to letters weighing more than 3.5 ounces.


Kathryn Sidenstricker / Dreamstime.com

*Source*: U.S. Postal Service Notice 123

(a) Find a function $C$ that models the first-class postage charged, in dollars, for a letter weighing $w$ ounces. Assume $w > 0$.

(b) What is the domain of $C$?

(c) Graph the function $C$.

(d) Use the graph to investigate $\lim_{w \to 2^-} C(w)$ and $\lim_{w \to 2^+} C(w)$. Do these suggest that $\lim_{w \to 2} C(w)$ exists?

(e) Use the graph to investigate $\lim_{w \to 0^+} C(w)$.

(f) Use the graph to investigate $\lim_{w \to 3.5^-} C(w)$.

**60. First-Class Mail** As of April 2016, the U.S. Postal Service charged \$0.94 postage for first-class large envelope weighing up to and including 1 ounce, plus a flat fee of \$0.21 for each additional or partial ounce up to and including 13 ounces. First-class rates do not apply to large envelopes weighing more than 13 ounces.

*Source*: U.S. Postal Service Notice 123

(a) Find a function $C$ that models the first-class postage charged, in dollars, for a large envelope weighing $w$ ounces. Assume $w > 0$.

(b) What is the domain of $C$?

(c) Graph the function $C$.

(d) Use the graph to investigate $\lim_{w \to 1^-} C(w)$ and $\lim_{w \to 1^+} C(w)$. Do these suggest that $\lim_{w \to 1} C(w)$ exists?

(e) Use the graph to investigate $\lim_{w \to 12^-} C(w)$ and $\lim_{w \to 12^+} C(w)$. Do these suggest that $\lim_{w \to 12} C(w)$ exists?

(f) Use the graph to investigate $\lim_{w \to 0^+} C(w)$.

(g) Use the graph to investigate $\lim_{w \to 13^-} C(w)$.

**61. Correlating Student Success to Study Time** Professor Smith claims that a student's final exam score is a function of the time $t$ (in hours) that the student studies. He claims that the closer to seven hours one studies, the closer to 100% the student scores on the final. He claims that studying significantly less than seven hours may cause one to be underprepared for the test, while studying significantly more than seven hours may cause "burnout."

(a) Write Professor Smith's claim symbolically as a limit.

(b) Write Professor Smith's claim using the $\varepsilon$-$\delta$ definition of limit.

*Source*: Submitted by the students of Millikin University.

**62.** The definition of the slope of the tangent line to the graph of $y = f(x)$ at the point $(c, f(c))$ is $m_{\tan} = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$.

Another way to express this slope is to define a new variable $h = x - c$. Rewrite the slope of the tangent line $m_{\tan}$ using $h$ and $c$.

**63.** If $f(2) = 6$, can you conclude anything about $\lim_{x \to 2} f(x)$? Explain your reasoning.

**64.** If $\lim_{x \to 2} f(x) = 6$, can you conclude anything about $f(2)$? Explain your reasoning.

**65.** The graph of $f(x) = \frac{x-3}{3-x}$ is a straight line with a point punched out.

(a) What straight line and what point?

(b) Use the graph of $f$ to investigate the one-sided limits of $f$ as $x$ approaches 3.

(c) Does the graph suggest that $\lim_{x \to 3} f(x)$ exists? If so, what is it?

**66.** (a) Use a table to investigate $\lim_{x \to 0}(1 + x)^{1/x}$.

(b) Use graphing technology to graph $g(x) = (1 + x)^{1/x}$.

(c) What do (a) and (b) suggest about $\lim_{x \to 0}(1 + x)^{1/x}$?

CAS (d) Find $\lim_{x \to 0}(1 + x)^{1/x}$.

## Challenge Problems

*For Problems 67–70, investigate each of the following limits.*

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is an integer} \\ 0 & \text{if } x \text{ is not an integer} \end{cases}$$

**67.** $\lim_{x \to 2} f(x)$ **68.** $\lim_{x \to 1/2} f(x)$ **69.** $\lim_{x \to 3} f(x)$ **70.** $\lim_{x \to 0} f(x)$

## AP® Practice Problems

PAGE 84 **1.** To investigate $\lim_{x \to 1^-} \sqrt{1 - x}$ using a table, evaluate $f(x) = \sqrt{1 - x}$, by choosing

(A) numbers close to 0, some slightly less than 0 and some slightly greater than 0.
(B) only numbers slightly less than 1.
(C) only numbers slightly greater than 1.
(D) numbers close to 1, some slightly less than 1 and some slightly greater than 1.

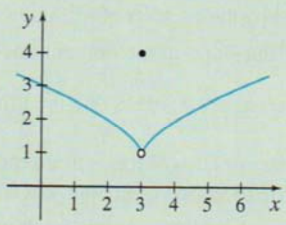
**2.** $\lim_{x \to -2^+} x^3 = -8$ is called a

(A) lower limit (B) negative limit
(C) positive limit (D) right-hand limit

PAGE 80 **3.** "The limit as $x$ approaches 0 of the function $f(x) = \cos x$ is equal to the number 1," is written symbolically as

(A) $\lim_{\cos x \to 0} \cos x = 1$ (B) $\lim_{x \to \cos x} \cos x = 1$
(C) $\lim_{x \to 0} \cos x = 1$ (D) $\lim_{x \to 1} \cos x = 0$

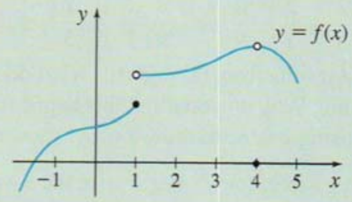
PAGE 83 **4.** The graph of a function $f$ is given below.




Which statement best describes $\lim_{x \to 3} f(x)$?

(A) $\lim_{x \to 3} f(x) = 1$ (B) $\lim_{x \to 3} f(x) = 3$
(C) $\lim_{x \to 3} f(x) = 4$ (D) $\lim_{x \to 3} f(x)$ doesn't exist.

PAGE 84 **5.** The graph of a piecewise function $f$ is shown.




Use the graph to determine which of the following statements is true.

(A) $\lim_{x \to -1^+} f(x)$ does not exist. (B) $\lim_{x \to 1^-} f(x) = f(1)$.
(C) $\lim_{x \to 1} f(x)$ exists. (D) $\lim_{x \to 4} f(x) = f(4)$.

PAGE 82 **6.** The table below gives values of three functions near 1.

| $x$ | 0.7 | 0.8 | 0.9 | 0.95 | 1 | 1.05 | 1.1 | 1.2 | 1.3 |
|---|---|---|---|---|---|---|---|---|---|
| $f(x)$ | 0 | 0 | 0 | 0 | 0 | 0.9 | 0.9 | 0.9 | 0.9 |
| $g(x)$ | −0.9 | −0.95 | −0.095 | −0.009 | undefined | 0.009 | 0.095 | 0.95 | 0.995 |
| $h(x)$ | 1.0 | 0.08 | 0.008 | 0.0008 | 1 | −0.005 | −0.025 | −0.05 | −0.25 |

For which of these functions does the table suggest that the limit as $x$ approaches 1 is 0?

(A) $f$ only (B) $h$ only (C) $f$ and $g$ only (D) $g$ and $h$ only

PAGE 80 **7.** Interpret $\lim_{x \to 2} (x^3 + 3x - 4) = 10$.

PAGE 82 **8.** Use a calculator to create a table to investigate $\lim_{x \to 0} \frac{e^x - 1}{x}$.

# 1.2 Limits of Functions Using Properties of Limits

**OBJECTIVES** *When you finish this section, you should be able to:*

**1** **Find the limit of a sum, a difference, and a product (p. 91)**
**2** **Find the limit of a power and the limit of a root (p. 94)**
**3** **Find the limit of a polynomial (p. 95)**
**4** **Find the limit of a quotient (p. 96)**
**5** **Find the limit of an average rate of change (p. 98)**
**6** **Find the limit of a difference quotient (p. 99)**

In Section 1.1, we used a numerical approach (tables) and a graphical approach to investigate limits. We saw that these approaches are not always reliable. The only way to be sure a limit is correct is to use the $\varepsilon$-$\delta$ definition of a limit. In this section, we state without proof results based on the $\varepsilon$-$\delta$ definition. Some of the results are proved in Section 1.6 and others in Appendix B.

We begin with two basic limits.