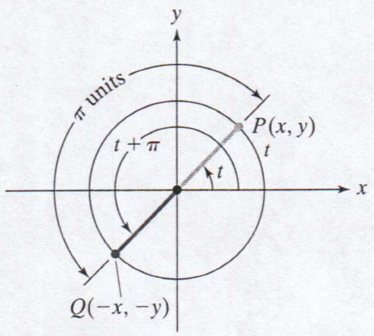


Figure 5.61 tangent at P = tangent at Q

Like the sine and cosine functions, the secant and cosecant functions have period 2π. However, the tangent and cotangent functions have a smaller period. **Figure 5.61** shows that if we begin at any point $P(x, y)$ on the unit circle and travel a distance of π units along the perimeter, we arrive at the point $Q(-x, -y)$. The tangent function, defined in terms of the coordinates of a point, is the same at (x, y) and $(-x, -y)$.

$$\underbrace{\frac{y}{x}}_{\text{Tangent function at } (x,y)} = \underbrace{\frac{-y}{-x}}_{\text{Tangent function } \pi \text{ radians later}}$$

We see that $\tan(t + \pi) = \tan t$. The same observations apply to the cotangent function.

Periodic Properties of the Tangent and Cotangent Functions

$$\tan(t + \pi) = \tan t \quad \text{and} \quad \cot(t + \pi) = \cot t$$

The tangent and cotangent functions are periodic functions and have period π.

Why do the trigonometric functions model phenomena that repeat *indefinitely*? By starting at point P on the unit circle and traveling a distance of 2π units, 4π units, 6π units, and so on, we return to the starting point P. Because the trigonometric functions are defined in terms of the coordinates of that point P, if we add (or subtract) multiples of 2π to t, the values of the trigonometric functions of t do not change. Furthermore, the values for the tangent and cotangent functions of t do not change if we add (or subtract) multiples of π to t.

Repetitive Behavior of the Sine, Cosine, and Tangent Functions

For any integer n and real number t,

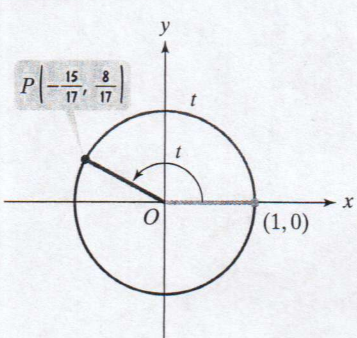
$$\sin(t + 2\pi n) = \sin t, \quad \cos(t + 2\pi n) = \cos t, \quad \text{and} \quad \tan(t + \pi n) = \tan t.$$

Exercise Set 5.4

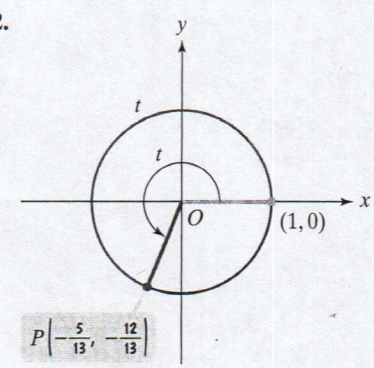
Practice Exercises

In Exercises 1–4, a point $P(x, y)$ is shown on the unit circle corresponding to a real number t. Find the values of the trigonometric functions at t.

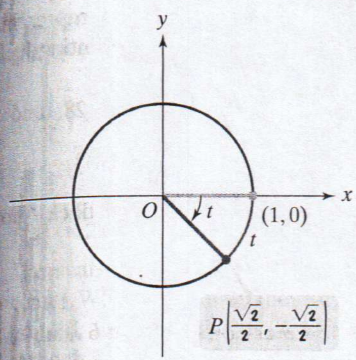
1.



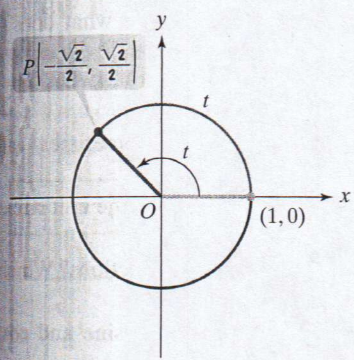
2.



3.



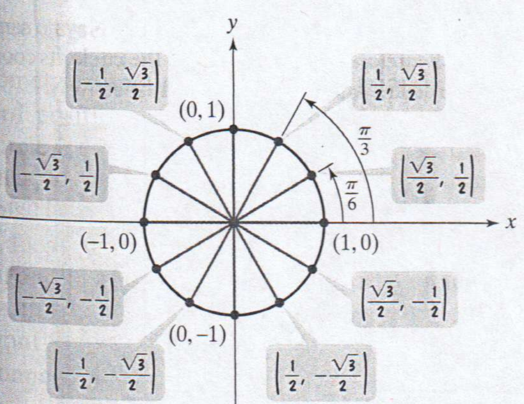
4.



In Exercises 5–18, the unit circle has been divided into twelve equal arcs, corresponding to t-values of

$$0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{4\pi}{3}, \frac{3\pi}{2}, \frac{5\pi}{3}, \frac{11\pi}{6}, \text{ and } 2\pi.$$

Use the (x, y) coordinates in the figure to find the value of each trigonometric function at the indicated real number, t, or state that the expression is undefined.



5. $\sin\frac{\pi}{6}$ **6.** $\sin\frac{\pi}{3}$ **7.** $\cos\frac{5\pi}{6}$

8. $\cos\frac{2\pi}{3}$ **9.** $\tan\pi$ **10.** $\tan 0$

11. $\csc\frac{7\pi}{6}$ **12.** $\csc\frac{4\pi}{3}$ **13.** $\sec\frac{11\pi}{6}$

14. $\sec\frac{5\pi}{3}$ **15.** $\sin\frac{3\pi}{2}$ **16.** $\cos\frac{3\pi}{2}$

17. $\sec\frac{3\pi}{2}$ **18.** $\tan\frac{3\pi}{2}$

In Exercises 19–24,

a. *Use the unit circle shown for Exercises 5–18 to find the value of the trigonometric function.*

b. *Use even and odd properties of trigonometric functions and your answer from part (a) to find the value of the same trigonometric function at the indicated real number.*

19. a. $\cos\frac{\pi}{6}$
b. $\cos\left(-\frac{\pi}{6}\right)$

20. a. $\cos\frac{\pi}{3}$
b. $\cos\left(-\frac{\pi}{3}\right)$

21. a. $\sin\frac{5\pi}{6}$
b. $\sin\left(-\frac{5\pi}{6}\right)$

22. a. $\sin\frac{2\pi}{3}$
b. $\sin\left(-\frac{2\pi}{3}\right)$

23. a. $\tan\frac{5\pi}{3}$
b. $\tan\left(-\frac{5\pi}{3}\right)$

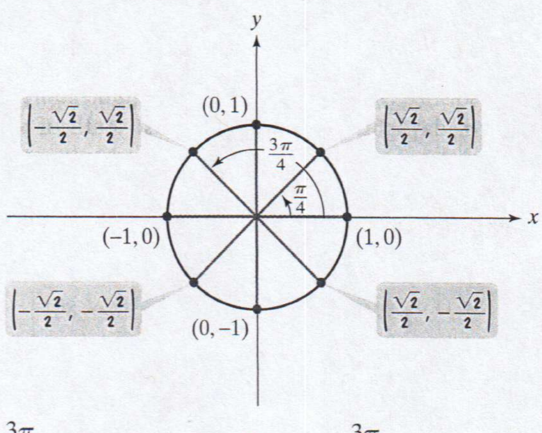
24. a. $\tan\frac{11\pi}{6}$
b. $\tan\left(-\frac{11\pi}{6}\right)$

In Exercises 25–32, the unit circle has been divided into eight equal arcs, corresponding to t-values of

$$0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}, \text{ and } 2\pi.$$

a. *Use the (x, y) coordinates in the figure to find the value of the trigonometric function.*

b. *Use periodic properties and your answer from part (a) to find the value of the same trigonometric function at the indicated real number.*



25. a. $\sin\frac{3\pi}{4}$
b. $\sin\frac{11\pi}{4}$

26. a. $\cos\frac{3\pi}{4}$
b. $\cos\frac{11\pi}{4}$

27. a. $\cos\frac{\pi}{2}$
b. $\cos\frac{9\pi}{2}$

28. a. $\sin\frac{\pi}{2}$
b. $\sin\frac{9\pi}{2}$

29. a. $\tan\pi$
b. $\tan 17\pi$

30. a. $\cot\frac{\pi}{2}$
b. $\cot\frac{15\pi}{2}$

31. a. $\sin\frac{7\pi}{4}$
b. $\sin\frac{47\pi}{4}$

32. a. $\cos\frac{7\pi}{4}$
b. $\cos\frac{47\pi}{4}$

Practice Plus

In Exercises 33–42, let

$$\sin t = a, \cos t = b, \text{ and } \tan t = c.$$

Write each expression in terms of a, b, and c.

33. $\sin(-t) - \sin t$
34. $\tan(-t) - \tan t$
35. $4\cos(-t) - \cos t$
36. $3\cos(-t) - \cos t$
37. $\sin(t + 2\pi) - \cos(t + 4\pi) + \tan(t + \pi)$
38. $\sin(t + 2\pi) + \cos(t + 4\pi) - \tan(t + \pi)$
39. $\sin(-t - 2\pi) - \cos(-t - 4\pi) - \tan(-t - \pi)$
40. $\sin(-t - 2\pi) + \cos(-t - 4\pi) - \tan(-t - \pi)$
41. $\cos t + \cos(t + 1000\pi) - \tan t - \tan(t + 999\pi) - \sin t + 4\sin(t - 1000\pi)$
42. $-\cos t + 7\cos(t + 1000\pi) + \tan t + \tan(t + 999\pi) + \sin t + \sin(t - 1000\pi)$

Application Exercises

43. The number of hours of daylight, H, on day t of any given year (on January 1, $t = 1$) in Fairbanks, Alaska, can be modeled by the function

$$H(t) = 12 + 8.3\sin\left[\frac{2\pi}{365}(t - 80)\right].$$

   a. March 21, the 80th day of the year, is the spring equinox. Find the number of hours of daylight in Fairbanks on this day.
   b. June 21, the 172nd day of the year, is the summer solstice, the day with the maximum number of hours of daylight. To the nearest tenth of an hour, find the number of hours of daylight in Fairbanks on this day.
   c. December 21, the 355th day of the year, is the winter solstice, the day with the minimum number of hours of daylight. Find, to the nearest tenth of an hour, the number of hours of daylight in Fairbanks on this day.

44. The number of hours of daylight, H, on day t of any given year (on January 1, $t = 1$) in San Diego, California, can be modeled by the function

$$H(t) = 12 + 2.4\sin\left[\frac{2\pi}{365}(t - 80)\right].$$

   a. March 21, the 80th day of the year, is the spring equinox. Find the number of hours of daylight in San Diego on this day.
   b. June 21, the 172nd day of the year, is the summer solstice, the day with the maximum number of hours of daylight. Find, to the nearest tenth of an hour, the number of hours of daylight in San Diego on this day.
   c. December 21, the 355th day of the year, is the winter solstice, the day with the minimum number of hours of daylight. To the nearest tenth of an hour, find the number of hours of daylight in San Diego on this day.

45. People who believe in biorhythms claim that there are three cycles that rule our behavior—the physical, emotional, and mental. Each is a sine function of a certain period. The function for our emotional fluctuations is

$$E = \sin\frac{\pi}{14}t,$$

where t is measured in days starting at birth. Emotional fluctuations, E, are measured from -1 to 1, inclusive, with 1 representing peak emotional well-being, -1 representing the low for emotional well-being, and 0 representing feeling neither emotionally high nor low.

   a. Find E corresponding to $t = 7$, 14, 21, 28, and 35. Describe what you observe.
   b. What is the period of the emotional cycle?

46. The height of the water, H, in feet, at a boat dock t hours after 6 A.M. is given by

$$H = 10 + 4\sin\frac{\pi}{6}t.$$

   a. Find the height of the water at the dock at 6 A.M., 9 A.M., noon, 6 P.M., midnight, and 3 A.M.
   b. When is low tide and when is high tide?
   c. What is the period of this function and what does this mean about the tides?

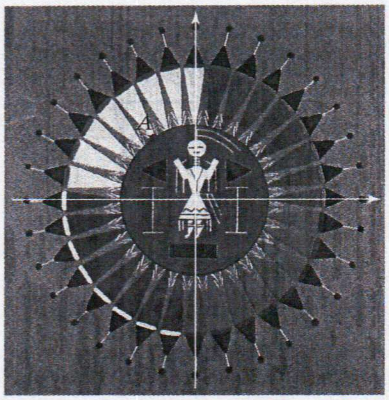
Writing in Mathematics

47. Why are the trigonometric functions sometimes called circular functions?
48. What is the range of the sine function? Use the unit circle to explain where this range comes from.
49. What do we mean by even trigonometric functions? Which of the six functions fall into this category?
50. What is a periodic function? Why are the sine and cosine functions periodic?
51. Explain how you can use the function for emotional fluctuations in Exercise 45 to determine good days for having dinner with your moody boss.
52. Describe a phenomenon that repeats infinitely. What is its period?

Critical Thinking Exercises

Make Sense? *In Exercises 53–56, determine whether each statement makes sense or does not make sense, and explain your reasoning.*

53. Assuming that the innermost circle on this Navajo sand painting is a unit circle, as A moves around the circle, its coordinates define the cosine and sine functions, respectively.



54. I'm using a value for t and a point on the unit circle corresponding to t for which $\sin t = -\frac{\sqrt{10}}{2}$.
55. Because $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$, I can conclude that $\cos\left(-\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}$.
56. I can find the exact value of $\sin\frac{7\pi}{3}$ using periodic properties of the sine function, or using a coterminal angle and a reference angle.

Check Point 9 A region that is 30° north of the Equator averages a minimum of 10 hours of daylight in December. Hours of daylight are at a maximum of 14 hours in June. Let x represent the month of the year, with 1 for January, 2 for February, 3 for March, and 12 for December. If y represents the number of hours of daylight in month x, use a sine function of the form $y = A\sin(Bx - C) + D$ to model the hours of daylight.

Exercise Set 5.5

Practice Exercises

In Exercises 1–6, determine the amplitude of each function. Then graph the function and $y = \sin x$ in the same rectangular coordinate system for $0 \le x \le 2\pi$.

1. $y = 4\sin x$ **2.** $y = 5\sin x$

3. $y = \frac{1}{3}\sin x$ **4.** $y = \frac{1}{4}\sin x$

5. $y = -3\sin x$ **6.** $y = -4\sin x$

In Exercises 7–16, determine the amplitude and period of each function. Then graph one period of the function.

7. $y = \sin 2x$ **8.** $y = \sin 4x$

9. $y = 3\sin\frac{1}{2}x$ **10.** $y = 2\sin\frac{1}{4}x$

11. $y = 4\sin\pi x$ **12.** $y = 3\sin 2\pi x$

13. $y = -3\sin 2\pi x$ **14.** $y = -2\sin\pi x$

15. $y = -\sin\frac{2}{3}x$ **16.** $y = -\sin\frac{4}{3}x$

In Exercises 17–30, determine the amplitude, period, and phase shift of each function. Then graph one period of the function.

17. $y = \sin(x - \pi)$ **18.** $y = \sin\left(x - \frac{\pi}{2}\right)$

19. $y = \sin(2x - \pi)$ **20.** $y = \sin\left(2x - \frac{\pi}{2}\right)$

21. $y = 3\sin(2x - \pi)$ **22.** $y = 3\sin\left(2x - \frac{\pi}{2}\right)$

23. $y = \frac{1}{2}\sin\left(x + \frac{\pi}{2}\right)$ **24.** $y = \frac{1}{2}\sin(x + \pi)$

25. $y = -2\sin\left(2x + \frac{\pi}{2}\right)$ **26.** $y = -3\sin\left(2x + \frac{\pi}{2}\right)$

27. $y = 3\sin(\pi x + 2)$ **28.** $y = 3\sin(2\pi x + 4)$

29. $y = -2\sin(2\pi x + 4\pi)$ **30.** $y = -3\sin(2\pi x + 4\pi)$

In Exercises 31–34, determine the amplitude of each function. Then graph the function and $y = \cos x$ in the same rectangular coordinate system for $0 \le x \le 2\pi$.

31. $y = 2\cos x$ **32.** $y = 3\cos x$

33. $y = -2\cos x$ **34.** $y = -3\cos x$

In Exercises 35–42, determine the amplitude and period of each function. Then graph one period of the function.

35. $y = \cos 2x$ **36.** $y = \cos 4x$

37. $y = 4\cos 2\pi x$ **38.** $y = 5\cos 2\pi x$

39. $y = -4\cos\frac{1}{2}x$ **40.** $y = -3\cos\frac{1}{3}x$

41. $y = -\frac{1}{2}\cos\frac{\pi}{3}x$ **42.** $y = -\frac{1}{2}\cos\frac{\pi}{4}x$

In Exercises 43–52, determine the amplitude, period, and phase shift of each function. Then graph one period of the function.

43. $y = \cos\left(x - \frac{\pi}{2}\right)$ **44.** $y = \cos\left(x + \frac{\pi}{2}\right)$

45. $y = 3\cos(2x - \pi)$ **46.** $y = 4\cos(2x - \pi)$

47. $y = \frac{1}{2}\cos\left(3x + \frac{\pi}{2}\right)$ **48.** $y = \frac{1}{2}\cos(2x + \pi)$

49. $y = -3\cos\left(2x - \frac{\pi}{2}\right)$ **50.** $y = -4\cos\left(2x - \frac{\pi}{2}\right)$

51. $y = 2\cos(2\pi x + 8\pi)$ **52.** $y = 3\cos(2\pi x + 4\pi)$

In Exercises 53–60, use a vertical shift to graph one period of the function.

53. $y = \sin x + 2$ **54.** $y = \sin x - 2$

55. $y = \cos x - 3$ **56.** $y = \cos x + 3$

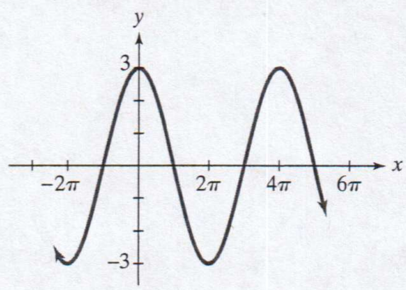
57. $y = 2\sin\frac{1}{2}x + 1$ **58.** $y = 2\cos\frac{1}{2}x + 1$

59. $y = -3\cos 2\pi x + 2$ **60.** $y = -3\sin 2\pi x + 2$

Practice Plus

In Exercises 61–66, find an equation for each graph.

61.



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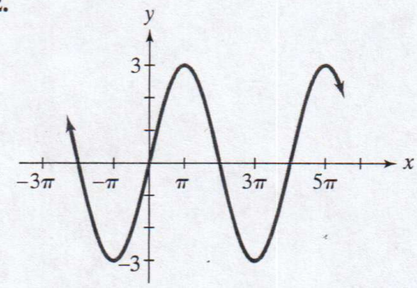

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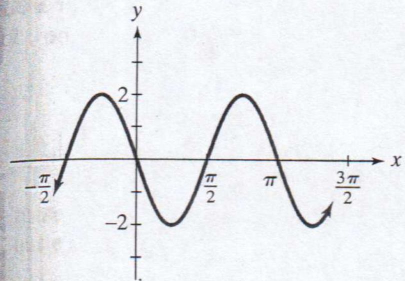
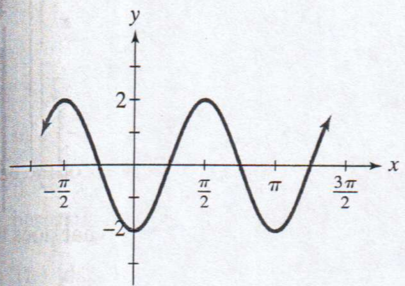


64.



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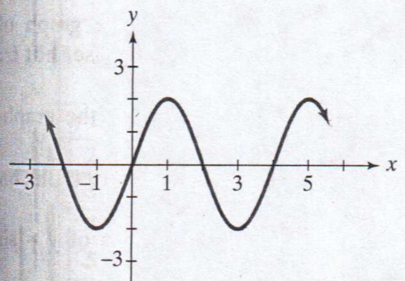
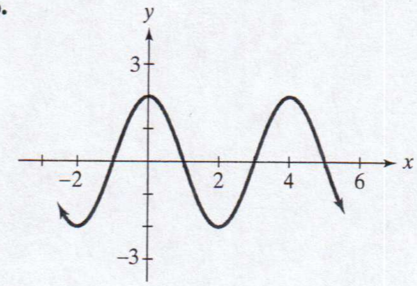


66.



In Exercises 67–70, graph one period of each function.

67. $y = \left|2 \cos \frac{x}{2}\right|$

68. $y = \left|3 \cos \frac{2x}{3}\right|$

69. $y = -|3 \sin \pi x|$

70. $y = -\left|2 \sin \frac{\pi x}{2}\right|$

In Exercises 71–74, graph f, g, and h in the same rectangular coordinate system for $0 \le x \le 2\pi$. Obtain the graph of h by adding or subtracting the corresponding y-coordinates on the graphs of f and g.

71. $f(x) = -2 \sin x$, $g(x) = \sin 2x$, $h(x) = (f + g)(x)$

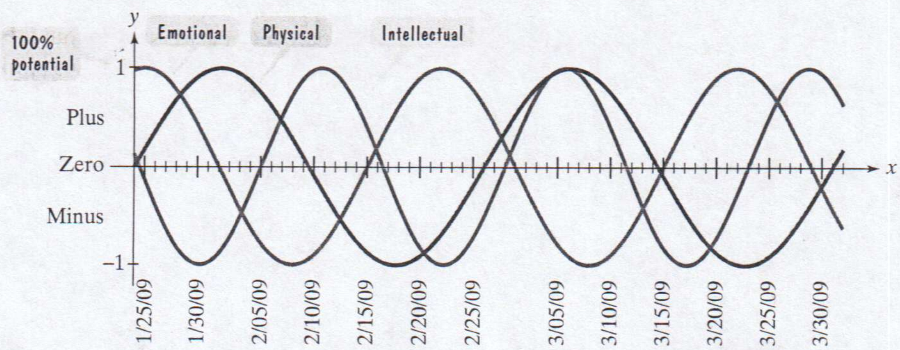
72. $f(x) = 2 \cos x$, $g(x) = \cos 2x$, $h(x) = (f + g)(x)$

73. $f(x) = \sin x$, $g(x) = \cos 2x$, $h(x) = (f - g)(x)$

74. $f(x) = \cos x$, $g(x) = \sin 2x$, $h(x) = (f - g)(x)$

Application Exercises

In the theory of biorhythms, sine functions are used to measure a person's potential. You can obtain your biorhythm chart online by simply entering your date of birth, the date you want your biorhythm chart to begin, and the number of months you wish to be included in the plot. Shown below is your author's chart, beginning January 25, 2009, when he was 23,283 days old. We all have cycles with the same amplitudes and periods as those shown here. Each of our three basic cycles begins at birth. Use the biorhythm chart shown to solve Exercises 75–82. The longer tick marks correspond to the dates shown.



75. What is the period of the physical cycle?

76. What is the period of the emotional cycle?

77. What is the period of the intellectual cycle?

78. For the period shown, what is the worst day in February for your author to run in a marathon?

79. For the period shown, what is the best day in March for your author to meet an online friend for the first time?

80. For the period shown, what is the best day in February for your author to begin writing this trigonometry chapter?

81. If you extend these sinusoidal graphs to the end of the year, is there a day when your author should not even bother getting out of bed?

82. If you extend these sinusoidal graphs to the end of the year, are there any days where your author is at near-peak physical, emotional, and intellectual potential?