Expressing an Exponential Model in Base *e*

$$y = ab^x \quad \text{is equivalent to} \quad y = ae^{(\ln b)\cdot x}.$$

EXAMPLE 6 Rewriting the Model for World Population in Base *e*

We have seen that the function

$$g(x) = 2.569(1.017)^x$$

models world population, $g(x)$, in billions, x years after 1949. Rewrite the model in terms of base e.

Solution We use the two equivalent equations shown in the voice balloons to rewrite the model in terms of base e.

$y = ab^x$ $\qquad\qquad$ $y = ae^{(\ln b)\cdot x}$

$$g(x) = 2.569(1.017)^x \quad \text{is equivalent to} \quad g(x) = 2.569e^{(\ln 1.017)\cdot x}.$$

Using $\ln 1.017 \approx 0.017$, the exponential growth model for world population, $g(x)$, in billions, x years after 1949 is

$$g(x) = 2.569e^{0.017x}.$$

In Example 6, we can replace $g(x)$ with A and x with t so that the model has the same letters as those in the exponential growth model $A = A_0e^{kt}$.

$A = A_o \; e^{kt}$ This is the exponential growth model.

$A = 2.569e^{0.017t}$ This is the model for world population.

The value of k, 0.017, indicates a growth rate of 1.7% per year. Although this is an excellent model for the data, we must be careful about making projections about world population using this growth function. Why? World population growth rate is now 1.2%, not 1.7%, so our model will overestimate future populations.

Check Point 6 Rewrite $y = 4(7.8)^x$ in terms of base e. Express the answer in terms of a natural logarithm and then round to three decimal places.

Exercise Set 4.5

Practice Exercises and Application Exercises

The exponential models describe the population of the indicated country, A, in millions, t years after 2006. Use these models to solve Exercises 1–6.

India $A = 1095.4e^{0.014t}$

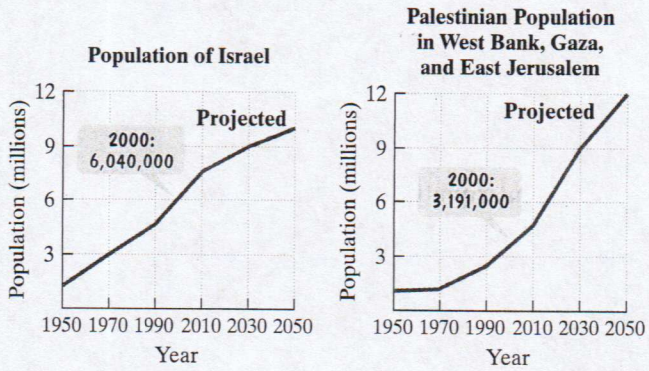
Iraq $A = 26.8e^{0.027t}$

Japan $A = 127.5e^{0.001t}$

Russia $A = 142.9e^{-0.004t}$

1. What was the population of Japan in 2006?
2. What was the population of Iraq in 2006?
3. Which country has the greatest growth rate? By what percentage is the population of that country increasing each year?
4. Which country has a decreasing population? By what percentage is the population of that country decreasing each year?
5. When will India's population be 1238 million?
6. When will India's population be 1416 million?

About the size of New Jersey, Israel has seen its population soar to more than 6 million since it was established. With the help of U.S. aid, the country now has a diversified economy rivaling those of other developed Western nations. By contrast, the Palestinians, living under Israeli occupation and a corrupt regime, endure bleak conditions. The graphs show that by 2050, Palestinians in the West Bank, Gaza Strip, and East Jerusalem will outnumber Israelis. Exercises 7–8 involve the projected growth of these two populations.



Source: Newsweek

7. a. In 2000, the population of Israel was approximately 6.04 million and by 2050 it is projected to grow to 10 million. Use the exponential growth model $A = A_0e^{kt}$, in which t is the number of years after 2000, to find an exponential growth function that models the data.

b. In which year will Israel's population be 9 million?

8. a. In 2000, the population of the Palestinians in the West Bank, Gaza Strip, and East Jerusalem was approximately 3.2 million and by 2050 it is projected to grow to 12 million. Use the exponential growth model $A = A_0e^{kt}$, in which t is the number of years after 2000, to find the exponential growth function that models the data.

b. In which year will the Palestinian population be 9 million?

In Exercises 9–14, complete the table. Round projected populations to one decimal place and values of k to four decimal places.

	Country	2007 Population (millions)	Projected 2025 Population (millions)	Projected Growth Rate, k
9.	Philippines	91.1		0.0147
10.	Pakistan	164.7		0.0157
11.	Colombia	44.4	55.2	
12.	Madagascar	19.4	32.4	
13.	South Africa	44.0	40.0	
14.	Bulgaria	7.3	6.3	

Source: International Programs Center, U.S. Census Bureau

An artifact originally had 16 grams of carbon-14 present. The decay model $A = 16e^{-0.000121t}$ describes the amount of carbon-14 present after t years. Use this model to solve Exercises 15–16.

15. How many grams of carbon-14 will be present in 5715 years?

16. How many grams of carbon-14 will be present in 11,430 years?

17. The half-life of the radioactive element krypton-91 is 10 seconds. If 16 grams of krypton-91 are initially present, how many grams are present after 10 seconds? 20 seconds? 30 seconds? 40 seconds? 50 seconds?

18. The half-life of the radioactive element plutonium-239 is 25,000 years. If 16 grams of plutonium-239 are initially present, how many grams are present after 25,000 years? 50,000 years? 75,000 years? 100,000 years? 125,000 years?

Use the exponential decay model for carbon-14, $A = A_0e^{-0.00012t}$, to solve Exercises 19–20.

19. Prehistoric cave paintings were discovered in a cave in France. The paint contained 15% of the original carbon-14. Estimate the age of the paintings.

20. Skeletons were found at a construction site in San Francisco in 1989. The skeletons contained 88% of the expected amount of carbon-14 found in a living person. In 1989, how old were the skeletons?

In Exercises 21–26, complete the table. Round half-lives to one decimal place and values of k to six decimal places.

	Radioactive Substance	Half-Life	Decay Rate, k
21.	Tritium		5.5% per year = −0.055
22.	Krypton-85		6.3% per year = −0.063
23.	Radium-226	1620 years	
24.	Uranium-238	4560 years	
25.	Arsenic-74	17.5 days	
26.	Calcium-47	113 hours	

27. The August 1978 issue of *National Geographic* described the 1964 find of bones of a newly discovered dinosaur weighing 170 pounds, measuring 9 feet, with a 6-inch claw on one toe of each hind foot. The age of the dinosaur was estimated using potassium-40 dating of rocks surrounding the bones.

a. Potassium-40 decays exponentially with a half-life of approximately 1.31 billion years. Use the fact that after 1.31 billion years a given amount of potassium-40 will have decayed to half the original amount to show that the decay model for potassium-40 is given by $A = A_0e^{-0.52912t}$, where t is in billions of years.

b. Analysis of the rocks surrounding the dinosaur bones indicated that 94.5% of the original amount of potassium-40 was still present. Let $A = 0.945A_0$ in the model in part (a) and estimate the age of the bones of the dinosaur.

Use the exponential decay model, $A = A_0e^{kt}$, to solve Exercises 28–31. Round answers to one decimal place.

28. The half-life of thorium-229 is 7340 years. How long will it take for a sample of this substance to decay to 20% of its original amount?

29. The half-life of lead is 22 years. How long will it take for a sample of this substance to decay to 80% of its original amount?

30. The half-life of aspirin in your bloodstream is 12 hours. How long will it take for the aspirin to decay to 70% of the original dosage?

31. Xanax is a tranquilizer used in the short-term relief of symptoms of anxiety. Its half-life in the bloodstream is 36 hours. How long will it take for Xanax to decay to 90% of the original dosage?

32. A bird species in danger of extinction has a population that is decreasing exponentially ($A = A_0e^{kt}$). Five years ago the population was at 1400 and today only 1000 of the birds are alive. Once the population drops below 100, the situation will be irreversible. When will this happen?

33. Use the exponential growth model, $A = A_0e^{kt}$, to show that the time it takes a population to double (to grow from A_0 to $2A_0$) is given by $t = \frac{\ln 2}{k}$.

34. Use the exponential growth model, $A = A_0e^{kt}$, to show that the time it takes a population to triple (to grow from A_0 to $3A_0$) is given by $t = \frac{\ln 3}{k}$.

Use the formula $t = \frac{\ln 2}{k}$ that gives the time for a population with a growth rate k to double to solve Exercises 35–36. Express each answer to the nearest whole year.

35. The growth model $A = 4.1e^{0.01t}$ describes New Zealand's population, A, in millions, t years after 2006.
    a. What is New Zealand's growth rate?
    b. How long will it take New Zealand to double its population?

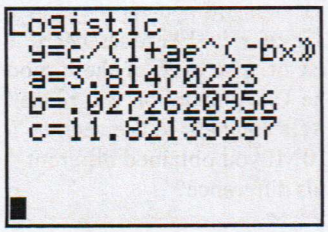
36. The growth model $A = 107.4e^{0.012t}$ describes Mexico's population, A, in millions, t years after 2006.
    a. What is Mexico's growth rate?
    b. How long will it take Mexico to double its population?

37. The logistic growth function

$$f(t) = \frac{100{,}000}{1 + 5000e^{-t}}$$

describes the number of people, $f(t)$, who have become ill with influenza t weeks after its initial outbreak in a particular community.
    a. How many people became ill with the flu when the epidemic began?
    b. How many people were ill by the end of the fourth week?
    c. What is the limiting size of the population that becomes ill?

Shown, again, at the top of the next column is world population, in billions, for seven selected years from 1950 through 2006. Using a graphing utility's logistic regression option, we obtain the equation shown on the screen.



x, Number of Years after 1949	y, World Population (billions)
1 (1950)	2.6
11 (1960)	3.0
21 (1970)	3.7
31 (1980)	4.5
41 (1990)	5.3
51 (2000)	6.1
57 (2006)	6.5

We see from the calculator screen that a logistic growth model for world population, $f(x)$, in billions, x years after 1949 is

$$f(x) = \frac{11.82}{1 + 3.81e^{-0.027x}}.$$

Use this function to solve Exercises 38–42.

38. How well does the function model the data for 2000?
39. How well does the function model the data for 2006?
40. When will world population reach 7 billion?
41. When will world population reach 8 billion?
42. According to the model, what is the limiting size of the population that Earth will eventually sustain?

The logistic growth function

$$P(x) = \frac{90}{1 + 271e^{-0.122x}}$$

models the percentage, $P(x)$, of Americans who are x years old with some coronary heart disease. Use the function to solve Exercises 43–46.

43. What percentage of 20-year-olds have some coronary heart disease?
44. What percentage of 80-year-olds have some coronary heart disease?
45. At what age is the percentage of some coronary heart disease 50%?
46. At what age is the percentage of some coronary heart disease 70%?

Exercises 47–50 present data in the form of tables. For each data set shown by the table,

a. *Create a scatter plot for the data.*
b. *Use the scatter plot to determine whether an exponential function or a logarithmic function is the best choice for modeling the data. (If applicable, in Exercise 70, you will use your graphing utility to obtain these functions.)*

47. **Percent of Miscarriages, by Age**

Woman's Age	Percent of Miscarriages
22	9%
27	10%
32	13%
37	20%
42	38%
47	52%

Source: Time